Néel walls with prescribed winding number and how a nonlocal term can change the energy landscape

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Abstract

We study a nonlocal Allen-Cahn type problem for vector fields of unit length, arising from a model for domain walls (called Néel walls) in ferromagnetism. We show that the nonlocal term gives rise to new features in the energy landscape; in particular, we prove existence of energy minimisers with prescribed winding number that would be prohibited in a local model.

Keywords: domain walls, Allen-Cahn, nonlocal, existence of minimizers, topological degree, concentration-compactness, micromagnetics

1 Introduction

1.1 Background

We study a model for one-dimensional transition layers, called Néel walls, that occur in thin ferromagnetic films. In the theory of micromagnetics, the magnetisation of a ferromagnetic sample is described by a vector field of unit length. In a typical model for Néel walls, the sample can be assumed to be two-dimensional and the vector field is tangential, which leads to a map with values in $\mathbb{S}^1$. We use a simplified model, also studied by several authors (see e.g. [2, 3, 4, 5, 6, 10, 12, 15, 16]), where it is assumed that the transition layers have a one-dimensional profile, described by a map $m\colon \mathbb{R} \to \mathbb{S}^1$. Our model is variational and the energy functional includes the Dirichlet integral, a multi-well potential, and a nonlocal term. The geometry of the problem allows us to define a topological degree (winding number) for the magnetisation that characterises the connected components of the relevant configuration space. Therefore, it is natural to study whether these connected components contain minimisers.

The corresponding problem for an Allen-Cahn type model (without a nonlocal term) is well understood: most connected components of the relevant space do not contain minimisers (see Appendix). We will show that the nonlocal term in our model changes the situation. In the simplest case, we will prove existence of minimisers with any prescribed winding number. We also study another case where a more intricate scenario aises: depending on a parameter, we have existence or nonexistence of minimisers for certain winding numbers.

1.2 The variational problem

We now describe the energy functional studied in this paper and the spaces where we look for minimisers. Our functional comprises three terms, coming from four different physical phenomena: magnetic anisotropy, an external magnetic field, the stray field generated by the magnetisation, and the quantum-mechanical spin interaction. The last of these gives rise to a term called exchange energy, which is modelled simply by the Dirichlet functional. The effects of the anisotropy and external field have the same general structure and are combined in effective anisotropy term in our model.

Anisotropy. Fix $h \geq 0$ with $h \neq 1$ and set $k = \min\{h, 1\} \in [0, 1]$. Define an anisotropy potential $W\colon \mathbb{S}^1 \to [0, \infty)$ by

$$W(m) = \frac{1}{2}(m_1^2 - 2hm_1 + 2hk - k^2) = \begin{cases} \frac{1}{2}(m_1 - k)^2 & \text{if } k = h < 1, \\ \frac{1}{2}(m_1 - k)^2 + (h-1)(1-m_1) & \text{if } k = 1 < h, \end{cases} \tag{1}$$

for $m = (m_1, m_2) \in \mathbb{S}^1$. If $h < 1$, then W has two wells on $\mathbb{S}^1$, at $(k, \pm\sqrt{1-k^2})$, while in the case $h > 1$, the potential W has one well on $\mathbb{S}^1$, at $(1, 0)$. In both situations, if we write $m = (\cos\theta, \sin\theta) \in \mathbb{S}^1$, then we have a pattern of periodically distributed wells in terms of the phase θ and W grows quadratically (in θ) near these wells (see Lemma 2.3). This behaviour is essential for our arguments and it is for this reason why we do not study the case $h = 1$ in this paper. In physical terms, W represents a combination of the micromagnetic anisotropy $m \mapsto m_1^2$, with easy axis parallel to the Néel walls, and an external magnetic field $h_{ext} = h\mathbf{e_1}$ perpendicular to the walls.

Stray field potential. Let

$$\mathbb{R}^2_+ = \mathbb{R} \times (0, \infty).$$

For a given map $m = (m_1, m_2)\colon \mathbb{R} \to \mathbb{S}^1$ such that $m_1 - k \in H^1(\mathbb{R})$, there exists a unique solution $u \in \dot{H}^1(\mathbb{R}^2_+)$, called the stray field potential, of the boundary value problem

$$\Delta u = 0 \qquad \text{in } \mathbb{R}^2_+, \tag{2}$$

$$\frac{\partial u}{\partial x_2} = -m_1' \quad \text{on } \mathbb{R} \times \{0\}, \tag{3}$$

where m_1' denotes the derivative of m_1. Here $\dot{H}^1(\mathbb{R}^2_+)$ denotes the completion of $C_0^\infty(\overline{\mathbb{R}^2_+})$ with respect to the inner product $\langle \cdot, \cdot \rangle_{\dot{H}^1(\mathbb{R}^2_+)}$, given by

$$\langle \phi, \psi \rangle_{\dot{H}^1(\mathbb{R}^2_+)} = \int_{\mathbb{R}^2_+} \nabla\phi \cdot \nabla\psi \, dx$$

for $\phi, \psi \in C_0^\infty(\overline{\mathbb{R}^2_+})$. Equivalently, u satisfies the identity

$$\int_{\mathbb{R}^2_+} \nabla u \cdot \nabla \zeta \, dx = \int_{-\infty}^{\infty} m_1' \zeta(\cdot, 0) \, dx_1 \quad \text{for every } \zeta \in C_0^\infty(\mathbb{R}^2), \tag{4}$$

where $x = (x_1, x_2)$. The elements of $\dot{H}^1(\mathbb{R}^2_+)$ are not functions (not even in the almost-everywhere sense), as the corresponding norm identifies all constants. But it is often convenient to treat them as functions nevertheless, while keeping the ambiguity in mind. The Dirichlet integral of u, called the stray field energy, can be computed in terms of the homogeneous $\|\cdot\|_{\dot{H}^{1/2}}$-seminorm of m_1, namely [9]

$$\frac{1}{2}\int_{\mathbb{R}^2_+} |\nabla u|^2 \, dx = \frac{1}{2}\int_{\mathbb{R}} \left| \left| \frac{d}{dx_1} \right|^{\frac{1}{2}} m_1 \right|^2 dx_1 = \tfrac{1}{2}\|m_1 - k\|^2_{\dot{H}^{1/2}}. \tag{5}$$

For a discussion of how this arises from micromagnetics, we refer to the work of DeSimone–Kohn–Müller–Otto [6].

Energy functional. We now define the functional E_h by the formula

$$E_h(m) = \frac{1}{2}\int_{-\infty}^{\infty} \left(|m'|^2 + 2W(m) \right) dx_1 + \frac{1}{2}\int_{\mathbb{R}^2_+} |\nabla u|^2 \, dx,$$

where $u \in \dot{H}^1(\mathbb{R}^2_+)$ is determined by (2) and (3). If $h < 1$, this is well-defined and finite for any $m \in H^1_{\mathrm{loc}}(\mathbb{R}; \mathbb{S}^1)$ such that $m_1 - k \in H^1(\mathbb{R})$ and $m_2' \in L^2(\mathbb{R})$. If $h > 1$, then we need to assume in addition that $m_1 - 1 \in L^1(\mathbb{R})$.

If $m \in H^1_{\mathrm{loc}}(\mathbb{R}; \mathbb{S}^1)$ with $E_h(m) < \infty$, then it is readily seen that the limits $\lim_{x_1 \to \pm\infty} m(x_1)$ exist and coincide with one of the zeros of W. That is, if $h > 1$, then

$$\lim_{x_1 \to \pm\infty} m_1(x_1) = (1, 0),$$

and if $h < 1$, then

$$\lim_{x_1 \to \pm\infty} m_1(x_1) = \left(h, \pm\sqrt{1-h^2}\right)$$

(where the signs on both sides of the equation are independent of one another). We choose

$$\alpha \in \left[0, \frac{\pi}{2}\right] \quad \text{such that } k = \cos\alpha.$$

(Thus $\alpha = 0$ if $h > 1$.)

Winding number. Let $m^\perp = (-m_2, m_1)$. It is easily seen that the quantity

$$\deg(m) = \frac{1}{2\pi} \int_{-\infty}^{\infty} m^\perp \cdot m' \, dx_1$$

exists and belongs to $\mathbb{Z} + \{0, \pm\frac{\alpha}{\pi}\}$ if $E_h(m) < \infty$. Moreover, this notion of topological degree (winding number) can be extended to all continuous maps $m \colon \mathbb{R} \to \mathbb{S}^1$ with $\lim_{x_1 \to \pm\infty} m_1(x_1) = k$. More precisely, for any such continuous map $m \colon \mathbb{R} \to \mathbb{S}^1$, there exists a continuous function $\phi \colon \mathbb{R} \to \mathbb{R}$, called the lifting of m, such that

$$m = (\cos\phi, \sin\phi) \quad \text{in } \mathbb{R}$$

and $\phi(\pm\infty) := \lim_{x_1 \to \pm\infty} \phi(x_1) \in 2\pi\mathbb{Z} + \{-\alpha, \alpha\}$. Our generalised winding number is given by

$$\deg(m) = \frac{\phi(+\infty) - \phi(-\infty)}{2\pi} \in \mathbb{Z} + \left\{0, \pm\frac{\alpha}{\pi}\right\}.$$

1.3 Main results

For any fixed $d \in \mathbb{Z} + \{0, \pm\frac{\alpha}{\pi}\}$, we define

$$\mathcal{A}_h(d) = \left\{m \in H^1_{\text{loc}}(\mathbb{R}; \mathbb{S}^1) \colon E_h(m) < \infty \text{ and } \deg(m) = d\right\} \tag{6}$$

and

$$\mathcal{E}_h(d) = \inf_{m \in \mathcal{A}_h(d)} E_h(m).$$

Note that $\{\mathcal{A}_h(d)\}_{d \in \mathbb{Z} + \{0, \pm\frac{\alpha}{\pi}\}}$ comprises the connected components of $\{m \in H^1_{\text{loc}}(\mathbb{R}; \mathbb{S}^1) \colon E_h(m) < \infty\}$ in the strong $\dot{H}^1(\mathbb{R})$ topology. (The map deg is continuous in the strong $\dot{H}^1(\mathbb{R})$ topology, thus every connected component is contained in one of the sets $\mathcal{A}_h(d)$. To see the connectedness of $\mathcal{A}_h(d)$, we consider the lifting: given two points in $\mathcal{A}_h(d)$, we may construct a path connection by interpolation of the corresponding liftings.) Thus it forms a partition of this set.

The following question is studied in this paper.

Question. Given $d \in \mathbb{Z} + \{0, \pm\frac{\alpha}{\pi}\}$, is $\mathcal{E}_h(d)$ attained? That is, does $m \in \mathcal{A}_h(d)$ exist such that $E_h(m) = \mathcal{E}_h(d)$?

The answer is clear for $d = 0$. Since for $m \in \mathcal{A}_h(d)$ (for any d), we can construct $\tilde{m}, \hat{m} \in \mathcal{A}_h(-d)$ by $\tilde{m}(x_1) = m(-x_1)$ and $\hat{m}_1 = m_1$, $\hat{m}_2 = -m_2$, it is also clear that the answer will always be the same for d and $-d$ (and that $\mathcal{E}_h(d) = \mathcal{E}_h(-d)$). Therefore, it suffices to consider $d > 0$.

In the case $h < 1$ and $d = \frac{\alpha}{\pi}$ or $d = 1 - \frac{\alpha}{\pi}$, the answer to the question is positive and was proved in the work of Chermisi-Muratov [2] (for $h = 0$, see also the work of Melcher [15]). In other words, if $h \in [0, 1)$, then $\mathcal{E}_h(\alpha/\pi)$ and $\mathcal{E}_h(1 - \alpha/\pi)$ are attained. These papers also give a lot of information about the structure of the minimisers. For $h > 1$ and $d = 1$, some of their arguments still work and give a positive answer. The underlying method relies on the symmetrisation of m_1 by rearrangements and the observation that the energy is decreased thereby. For higher winding numbers, the situation is more complicated and requires different arguments.

Our first main result shows that we have energy minimisers of any admissible winding number if $h > 1$. They correspond to arrays of Néel walls as observed in physical experiments [8, Fig. 5.66].

Theorem 1.1. *Suppose that $h > 1$. Then $\mathcal{E}_h(d)$ is attained for any $d \in \mathbb{Z}$.*

In contrast, for $h < 1$, we sometimes have a negative answer. In particular, we do not have any minimisers of winding number 1.[1]

Theorem 1.2. *If $h \in [0, 1)$, then $\mathcal{E}_h(1) = \mathcal{E}_h(\alpha/\pi) + \mathcal{E}_h(1 - \alpha/\pi)$ and $\mathcal{E}_h(1)$ is* not *attained.*

In general, the case $h < 1$ is much more subtle than $h > 1$, because the nonlocal term $\frac{1}{2}\int_{\mathbb{R}^2_+} |\nabla u|^2 \, dx$ in the energy gives rise simultaneously to attractive and repulsive interactions between different parts of the profile of m. We have only partial results here, but we do know the following.

Theorem 1.3. *There exists $H \in (0, 1)$ such that $\mathcal{E}_h(2 - \alpha/\pi)$ is attained whenever $h \in [H, 1)$.*

[1]According to the anonymous referee, for $h = 0$, the non-existence result for minimisers of winding number 1 was announced by Capella, Knüpfer, and Muratov at the Workshop on Micromagnetics: Analysis and Applications, University of Heidelberg, August 2014. However, to our knowledge, no such result has been published as of today (June 23, 2017).

We also prove the following Pohozaev identity for every critical point m of our energy, expressing equality of the exchange energy and the anisotropy energy.

Proposition 1.1. *Let $m : \mathbb{R} \to \mathbb{S}^1$ be a critical point of E_h with $E_h(m) < \infty$. Then*

$$\int_{\mathbb{R}} |m'|^2 \, dx_1 = 2 \int_{\mathbb{R}} W(m) \, dx_1.$$

We will also prove some qualitative and quantitative properties of the minimizers of E_h: symmetry properties (see Lemma 3.2 below) and decay rates at infinity that are exponential in the case of $h > 1$ and polynomial if $h < 1$, respectively (see Theorems 5.1, 5.2 and 5.3 below).

1.4 Heuristics

The key to the proofs of our results is control of the nonlocal energy. For this purpose, we need to understand the shape of energy minimising profiles m. A prescribed winding number d gives rise to a certain number of transitions of m between the wells of the anisotropy potential W. Each of these transitions represents a Néel wall (to use the micromagnetics jargon). In the case of $h > 1$, we have 2π-Néel walls, while for $1 > h = \cos\alpha$ (with $\alpha \in (0, \frac{\pi}{2}]$), we have Néel walls of angle 2α and $2\pi - 2\alpha$, respectively (see Figure 1).

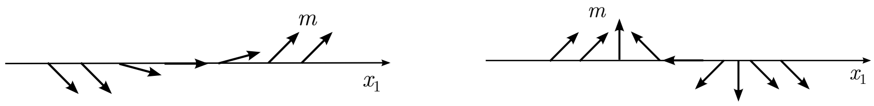

Figure 1: Schematic representation of a Néel wall of angle 2α (left) and $2\pi - 2\alpha$ (right).

In terms of the m_1 component, we can distinguish these two types of walls as follows: if $h < 1$, then a wall of angle 2α entails that m_1 attains the value 1 somewhere during the transition and we expect that m_1 exceeds $\cos\alpha$ throughout, while for a wall of angle $2\pi - 2\alpha$, we expect that m_1 is below $\cos\alpha$ and attains -1 at some point. For $h > 1$ (i.e., when W has a single well at $(1, 0)$), only the second alternative can occur (see Figure 2).

Our first observation is that the stray field energy will give rise to attraction between pairs of walls where $m_1 - \cos\alpha$ has the same sign, and repulsion otherwise. In particular, in the case $h > 1$, we only have attraction. We will prove that this effect of the nonlocal energy term dominates the interaction coming from the local energy terms.

As our energy controls the H^1-norm of m, the only possible cause for lack of compactness is escape to infinity of some walls. We can rule this out, using the previously described attraction, in the following cases.

(i) If $h > 1$, only attraction is possible; this is the situation in Theorem 1.1 (see also Figure 2).

(ii) If $h < 1$, the attraction between the outermost walls may be strong enough to keep the whole profile together. This is the case in Theorem 1.3 where a small wall is "sandwiched" between two large walls (see Figure 3, right).

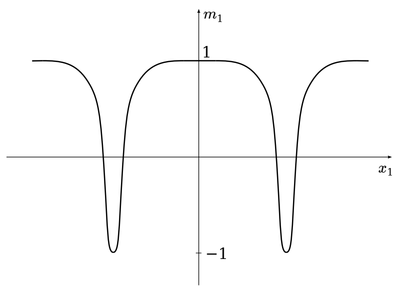

Figure 2: For $h > 1$, a pair of Néel walls of total winding number 2, represented in terms of m_1.

On the other hand, if one of the outermost walls is small relative to the adjoining one (or of comparable size), then there will be a strong repulsion that cannot be compensated by the remaining profile (as it is further away), in which case we expect nonexistence (see Figure 3, left). We prove this when $h < 1$ and the winding number is one (see Theorem 1.2).

In the remaining cases, we do not have a proof yet, but the following behaviour seems plausible.

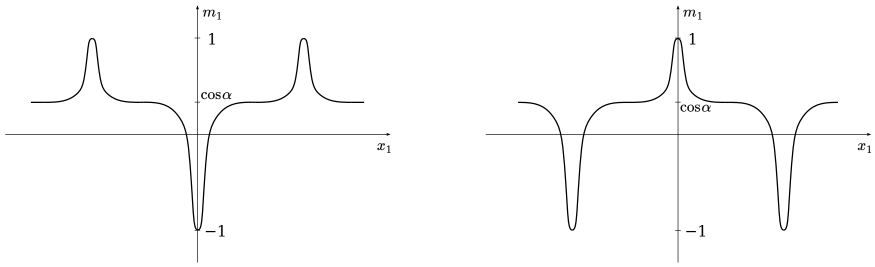

Figure 3: For $h < 1$, a hypothetical array of Néel walls of total winding number $1 + \alpha/\pi$ (left) and an existing one of winding number $2 - \alpha/\pi$ (right).

Conjecture 1.1. *If $h \in [0, 1)$, then for any $d \in \mathbb{N} = \{1, 2, \dots\}$, neither $\mathcal{E}_h(d)$ nor $\mathcal{E}_h(d + \alpha/\pi)$ are attained.*

Conjecture 1.2. *For any $d \in \mathbb{N}$, there exists $H \in (0, 1)$ such that $\mathcal{E}_h(d - \alpha/\pi)$ is attained whenever $h \in [H, 1)$.*

Conjecture 1.3. *For any $d \in \mathbb{N} \setminus \{1\}$, there exists $K \in (0, 1)$ such that $\mathcal{E}_h(d - \alpha/\pi)$ is* not *attained whenever $h \in [0, K]$.*

1.5 Other representations of the energy and the winding number

It is sometimes convenient to represent the energy functional E_h in terms of a phase (lifting) ϕ of m such that $m = (\cos\phi, \sin\phi)$. Abusing notation and writing $W(\phi)$ and $E_h(\phi)$ instead of $W(m)$ and $E_h(m)$, respectively, we have

$$E_h(\phi) = \frac{1}{2} \int_{-\infty}^{\infty} \left((\phi')^2 + 2W(\phi) \right) dx_1 + \frac{1}{2} \int_{\mathbb{R}^2_+} |\nabla u|^2 \, dx.$$

By definition, the potential W depends only on m_1, so we abuse notation further and write $W(m_1)$ instead of $W(m)$ when convenient. Since the stray field energy is determined by m_1 as well, we can rewrite the energy E_h in terms of m_1 only:

$$E_h(m) = \frac{1}{2} \int_{-\infty}^{\infty} \left(\frac{(m_1')^2}{1 - m_1^2} + 2W(m_1) \right) dx_1 + \frac{1}{2} \int_{\mathbb{R}} \left| \left| \frac{d}{dx_1} \right|^{\frac{1}{2}} m_1 \right|^2 dx_1.$$

In fact, often it is convenient to study our variational problem in terms of m_1 only, ignoring the second component m_2. Then we note that the winding number is characterised implicitly by the following simple observation.

Lemma 1.1. *Let $d \in \mathbb{N} + \{0, \pm\frac{\alpha}{\pi}\} \cup \{\frac{\alpha}{\pi}\}$. Let $m_1 \colon \mathbb{R} \to [-1, 1]$ be a continuous function with $\lim_{x_1 \to \pm\infty} m_1(x_1) = k$. Suppose that there exist $a_1, \dots, a_I \in \mathbb{R}$ with $a_1 < a_2 < \cdots < a_I$ and there exists $\epsilon \in \{\pm 1\}$ such that $m_1(a_j) = \epsilon(-1)^j$ for $j = 1, \dots, I$. Further suppose that one of the following conditions is satisfied:*

(i) I is odd and $d = \frac{I+1}{2} - \frac{\alpha}{\pi}$ and $\epsilon = 1$; or

(ii) I is odd and $d = \frac{I-1}{2} + \frac{\alpha}{\pi}$ and $\epsilon = -1$ and $h < 1$; or

(iii) I is even and $d = \frac{I}{2}$ and $h < 1$.

Then there exists a continuous function $m_2 \colon \mathbb{R} \to [-1, 1]$ such that the map $m = (m_1, m_2)$ takes values in $\mathbb{S}^1$ and $\deg(m) = d$.

Proof. We only give the arguments under the condition (i), as the proof is similar for the other cases. Since we need to satisfy $m_1^2 + m_2^2 = 1$ everywhere, we only need to determine the sign of m_2. Assuming that (i) is satisfied, we can do this as follows: in $(-\infty, a_1)$, we choose $m_2 = \sqrt{1 - m_1^2}$; in $[a_j, a_{j+1})$, we choose $m_2 = (-1)^j \sqrt{1 - m_1^2}$ for $j = 1, \dots, I - 1$; and in $[a_I, \infty)$, we choose $m_2 = -\sqrt{1 - m_1^2}$. This clearly gives rise to $m = (m_1, m_2)$ with the desired winding number. $\square$

1.6 Notation

The stray field potential $U(m)$. Recalling the Neumann problem (2)–(3) for $m_1 - k \in H^1(\mathbb{R})$, we highlight that the solutions u in $\dot{H}^1(\mathbb{R}^2_+)$ have a limit for $|x| \to \infty$. Indeed, if we extend u to $\mathbb{R}^2$ by even reflection, then we obtain a harmonic function near ∞ with finite Dirichlet energy, and it is well-known that the limit exists at ∞. Then we normalise this constant and define $U(m)$ (sometimes also denoted $U(m_1)$) to be the unique solution of (4) in $\dot{H}^1(\mathbb{R}^2_+)$ with

$$U(m) \to 0 \quad \text{as } |x| \to \infty.$$

If we denote the Fourier transform with respect to x_1 by $\mathcal{F}$, then the solution $U(m)$ is given by [9, Proposition 4]

$$\mathcal{F}U(m)(\xi, x_2) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x_1} U(m)(x)\, dx_1 = \frac{e^{-|\xi| x_2}}{|\xi|} \mathcal{F}(m_1')(\xi), \quad \xi \in \mathbb{R},\ x_2 \geq 0. \tag{7}$$

Note that $U(m) \in L^2(\mathbb{R}^2_+)$ if, and only if, $m_1 - k \in \dot{H}^{-1/2}(\mathbb{R})$, where the homogeneous Sobolev space $\dot{H}^s(\mathbb{R})$ (for $s \in \mathbb{R}$) is the set of tempered distributions f such that $\mathcal{F}f \in L^1_{\text{loc}}(\mathbb{R})$ and

$$\|f\|^2_{\dot{H}^s(\mathbb{R})} = \int_{\mathbb{R}} |\xi|^{2s} |\mathcal{F}f|^2\, d\xi < \infty.$$

The conjugate harmonic potential $V(m)$. In addition, we consider the conjugate harmonic function $V(m) \in \dot{H}^1(\mathbb{R}^2_+)$ (sometimes also denoted $V(m_1)$) with

$$\nabla^\perp V(m) = -\nabla U(m) \quad \text{in } \mathbb{R}^2_+.$$

In other words, $V(m)$ is the unique solution of the Dirichlet problem

$$\Delta V(m) = 0 \quad \text{in } \mathbb{R}^2_+, \tag{8}$$
$$V(m) = m_1 - k \quad \text{on } \mathbb{R} \times \{0\}. \tag{9}$$

Equivalently, $V(m)$ is the unique minimiser for the problem

$$\int_{\mathbb{R}^2_+} |\nabla V(m)|^2\, dx = \inf \left\{ \int_{\mathbb{R}^2_+} |\nabla v|^2\, dx \colon v \in \dot{H}^1(\mathbb{R}^2_+) \text{ with (9)} \right\}.$$

It is given by the following formula, similar to (7) [9, Proposition 3]:

$$\mathcal{F}V(m)(\xi, x_2) = e^{-|\xi| x_2} \mathcal{F}(m_1 - k)(\xi), \quad \xi \in \mathbb{R},\ x_2 \geq 0. \tag{10}$$

As $\mathcal{F}\left(x_1 \to \frac{x_2}{x_1^2 + x_2^2}\right)(\xi) = \sqrt{\frac{\pi}{2}} e^{-x_2|\xi|}$, we deduce the following Poisson formula:

$$V(m)(x) = \frac{x_2}{\pi} \int_{\mathbb{R}} \frac{m_1(t) - k}{(t - x_1)^2 + x_2^2}\, dt, \quad x \in \mathbb{R}^2_+.$$

The Dirichlet-to-Neumann operator Λ. Consider the operator $\Lambda \colon \dot{H}^1(\mathbb{R}) \to L^2(\mathbb{R})$ given by

$$\Lambda : f \mapsto -\left(-\frac{d^2}{dx_1^2}\right)^{\frac{1}{2}} f, \quad \text{i.e.,} \quad \mathcal{F}(\Lambda f)(\xi) = -|\xi| \mathcal{F}f(\xi), \quad \xi \in \mathbb{R}.$$

We can represent Λ by the following formula [7, (3.1)]:

$$\Lambda f(x_1) = \frac{1}{\pi} \operatorname{PV} \int_{-\infty}^{\infty} \frac{f(t) - f(x_1)}{(t - x_1)^2}\, dt, \quad x_1 \in \mathbb{R}. \tag{11}$$

By (7) and (10), we obtain

$$\Lambda(m_1 - k)(x_1) = \frac{\partial}{\partial x_1} U(m)(x_1, 0) = \frac{\partial}{\partial x_2} V(m)(x_1, 0), \quad x_1 \in \mathbb{R}. \tag{12}$$

Therefore, this is a Dirichlet-to-Neumann operator for the boundary value problem (8)–(9). If $u = U(m)$, we will often write u' for the quantity $u'(x_1) = \frac{\partial}{\partial x_1} U(m)(x_1, 0)$, where $x_1 \in \mathbb{R}$.

Remark 1.1. The Dirichlet-to-Neumann operator can be also defined on the space $\dot{H}^{1/2}(\mathbb{R})$, such that $\Lambda \colon \dot{H}^{1/2}(\mathbb{R}) \to \dot{H}^{-1/2}(\mathbb{R})$. Moreover, it is not difficult to see that

$$\Lambda\left(H^1_{\text{loc}} \cap L^\infty \cap \dot{H}^{1/2}(\mathbb{R})\right) \subset L^2_{\text{loc}} \cap \dot{H}^{-1/2}(\mathbb{R}).$$

Convention. Throughout the paper, when we speak of a *universal constant*, we mean a constant that depends neither on the parameter h nor on any of the variables of the problem.

1.7 Organisation of the paper

The rest of the paper is devoted to the proofs of our results. We first prove a few auxiliary statements in Sect. 2. Among these are estimates for $\mathcal{E}_h$, a proof that $W(\phi)$ grows quadratically in the phase ϕ near its zeros, and estimates of the energy for a profile localised with a cut-off function.

In Sect. 3, we state the Euler-Lagrange equation for critical points of E_h and a regularity result. We prove Proposition 1.1 here and we establish further consequences of the Euler-Lagrange equation, in particular a result on the symmetry of minimisers and H^2-estimates.

As the control of the nonlocal part of the energy is crucial for our analysis, we study this term in Sect. 4. We derive several estimates based on cut-off arguments and we establish the attraction/repulsion described in Sect. 1.4.

In Sect. 5, we analyse the tails of energy minimisers and their decay as $x_1 \to \pm\infty$. For $h > 1$, we obtain exponential decay. For $h < 1$, we can expect polynomial decay at best, and we prove this for winding numbers α/π and $1 - \alpha/\pi$ with the help of a linearisation of the Euler-Lagrange equation. These estimates are important in order to see that the attraction or repulsion of the nonlocal terms dominates everything else.

In Sect. 6, we establish a general concentration-compactness result that allows to prove existence of minimisers by finding good estimates for the energy. Finally, in Sect. 7, we combine all the ingredients and prove Theorems 1.1–1.3. In order to compare our results with the situation for a similar functional without a nonlocal term, we discuss the known results for the latter in the Appendix.

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2 Preliminary observations

2.1 A simple energy estimate

Suppose that $h \in [0, 1)$ and we study $\mathcal{E}_h(\alpha/\pi)$. While the work of Chermisi-Muratov [2] gives a lot of information about this situation (especially concerning the structure of the energy minimisers), we also need to know how $\mathcal{E}_h(\alpha/\pi)$ depends on α (and therefore on h). In particular the growth behaviour in α near 0 is important, e.g., for the proof of Theorem 1.3. An estimate can be obtained by a scaling argument as follows.

Lemma 2.1 (Cubic growth in α)**.** *There exists a universal constant $C > 0$ such that for all $h \in [0, 1)$,*

$$\mathcal{E}_h(\alpha/\pi) \leq C\alpha^3,$$

where $\alpha \in (0, \frac{\pi}{2}]$ with $\cos\alpha = h$.

Proof. Choose an increasing, smooth function $\tilde{\phi} : \mathbb{R} \to \mathbb{R}$ such that $\lim_{x_1\to\pm\infty} \tilde{\phi}(x_1) = \pm\pi/2$ and $\tilde{m} = (\cos\tilde{\phi}, \sin\tilde{\phi}) \in H^1(\mathbb{R}; \mathbb{S}^1)$. Let $\tilde{u} = U(\tilde{m})$ as defined in (7). Note that $\tilde{m} \in \mathcal{A}_0(1/2)$ according to the notation introduced in (6). Now define

$$\hat{m}_1 = 1 - (1 - \cos\alpha)(1 - \tilde{m}_1).$$

Then there exists a function $\hat{m}_2 : \mathbb{R} \to [-1, 1]$ such that $\hat{m} = (\hat{m}_1, \hat{m}_2) \in \mathcal{A}_h(\alpha/\pi)$. Let $\hat{u} = U(\hat{m})$. We compute

$$\int_{\mathbb{R}^2_+} |\nabla\hat{u}|^2\, dx = (1 - \cos\alpha)^2 \int_{\mathbb{R}^2_+} |\nabla\tilde{u}|^2\, dx$$

and

$$\int_{-\infty}^{\infty} (\hat{m}_1 - \cos\alpha)^2\, dx_1 = (1 - \cos\alpha)^2 \int_{-\infty}^{\infty} \tilde{m}_1^2\, dx_1.$$

Moreover, we have

$$1 - \hat{m}_1 = (1 - \cos\alpha)(1 - \tilde{m}_1),$$

while

$$1 + \hat{m}_1 \geq 1 + \tilde{m}_1.$$

Hence

$$\int_{-\infty}^{\infty} |\hat{m}'|^2\, dx_1 = \int_{-\infty}^{\infty} \frac{(\hat{m}_1')^2}{1 - \hat{m}_1^2}\, dx_1 \leq (1 - \cos\alpha) \int_{-\infty}^{\infty} \frac{(\tilde{m}_1')^2}{1 - \tilde{m}_1^2}\, dx_1.$$

Finally, let $m(x_1) = \hat{m}\left(x_1\sqrt{1-\cos\alpha}\right)$. Then it follows that

$$E_h(m) \leq (1-\cos\alpha)^{3/2} E_0(\tilde{m}),$$

which implies the desired inequality. $\square$

For the transition angle $1-\alpha/\pi$, we have the following uniform energy estimate.

Lemma 2.2. *There exists a universal constant C such that for all $h \in [0,1)$ with $\alpha = \arccos h \in (0, \frac{\pi}{2}]$,*

$$\mathcal{E}_h(1-\alpha/\pi) \leq C.$$

Proof. Choose $\eta \in C^\infty(\mathbb{R})$ with $\eta \equiv 0$ in $(-\infty,-1]$ and $\eta \equiv 1$ in $[1,\infty)$. Define $\phi = \alpha + (2\pi - 2\alpha)\eta$ and $m = (\cos\phi, \sin\phi)$. Then it is clear that $m \in \mathcal{A}_h(1-\alpha/\pi)$ and

$$\|m'\|_{L^2(\mathbb{R})} = \|\phi'\|_{L^2(\mathbb{R})} \leq 2\pi\|\eta'\|_{L^2(\mathbb{R})}.$$

Furthermore, as $\operatorname{supp}(m_1 - h) \subset [-1,1]$, we have $\|m_1 - h\|_{L^2(\mathbb{R})} \leq 2\sqrt{2}$. By standard interpolation inequalities, we obtain a uniform estimate for $\|m_1 - h\|_{\dot{H}^{1/2}(\mathbb{R})}$ as well, and the claim follows. $\square$

2.2 Behaviour of the anisotropy W near its zeros

The function $\phi \mapsto W(\cos\phi, \sin\phi)$ grows quadratically near its zeros. This behaviour is crucial for our analysis and we will need the following estimates.

Lemma 2.3. *There exists a universal constant $\gamma > 0$ such that for all $m = (\cos\phi, \sin\phi) \in \mathbb{S}^1$ with $\phi \in [-\pi,\pi]$, the following inequalities hold true. If $h \in [0,1)$ with $\alpha = \arccos h \in (0, \frac{\pi}{2}]$, then*

$$W(m) \geq \gamma^2(\phi^2 - \alpha^2)^2.$$

If $h > 1$, then

$$W(m) \geq (h-1)(1-\cos\phi) \geq (h-1)\gamma^2\phi^2.$$

Proof. Suppose first that $h \in [0,1)$. Define the function $w\colon \mathbb{R}^2 \to \mathbb{R}$ by $w(\phi,\alpha) = \frac{1}{2}(\cos\phi - \cos\alpha)^2$ and note that $W(m) = w(\phi,\alpha)$ when $m = (\cos\phi, \sin\phi)$. The function w is smooth with vanishing derivatives up to third order at $(0,0)$. Moreover, we compute

$$\frac{\partial^4 w}{\partial\phi^4}(0,0) = 3, \quad \frac{\partial^4 w}{\partial\phi^3\partial\alpha}(0,0) = 0, \quad \frac{\partial^4 w}{\partial\phi^2\partial\alpha^2}(0,0) = -1, \quad \frac{\partial^4 w}{\partial\phi\partial\alpha^3}(0,0) = 0, \quad \frac{\partial^4 w}{\partial\alpha^4}(0,0) = 3.$$

Therefore, by Taylor's theorem, we have

$$\lim_{(\phi,\alpha)\to(0,0)} \frac{w(\phi,\alpha)}{(\phi^2-\alpha^2)^2} = \frac{1}{8}.$$

Similarly, we see that for any $\alpha \in (0, \frac{\pi}{2}]$,

$$\lim_{\phi\to\pm\alpha} \frac{w(\phi,\alpha)}{(\phi^2-\alpha^2)^2} = \frac{\sin^2\alpha}{8\alpha^2}.$$

This implies that the function $(\phi,\alpha) \mapsto w(\phi,\alpha)/(\phi^2-\alpha^2)^2$ has a continuous, positive extension to $[-\pi,\pi] \times [0, \frac{\pi}{2}]$. By the compactness of this domain, the claim follows in this case.

Now suppose that $h > 1$. Then $W(m) = (h-1)(1-\cos\phi) + \frac{1}{2}(\cos\phi - 1)^2$. As there exists a number $c > 0$ such that $1 - \cos\phi \geq c\phi^2$ for every $\phi \in [-\pi,\pi]$, the desired inequality follows. $\square$

2.3 Localisation

For minimisers m of E_h, the function $m_1 - k$ will decay at a certain rate as $x_1 \to \pm\infty$, as we will eventually see. This will allow us to replace m by a map $\tilde{m}$ such that $\tilde{m}_1 - k$ has support in a bounded interval, while changing the energy by only a small amount. Quantifying this amount is also essential for the proof of existence of minimizers in our main results. More precisely, we have the following.

Proposition 2.1. *There exists a constant $C > 0$ with the following property. Suppose that $\phi \in H^1_{\mathrm{loc}}(\mathbb{R})$ is such that $m = (\cos\phi, \sin\phi)$ satisfies $E_h(m) < \infty$. Furthermore, suppose that there exist two numbers $\ell_\pm \in 2\pi\mathbb{Z} + \{-\alpha, \alpha\}$ and three measurable functions $\omega, \sigma, \tau \colon [0,\infty) \to (0,\infty)$ such that*

$$|\phi(x_1) - \ell_+| \le \omega(x_1) \quad \text{and} \quad |\phi(-x_1) - \ell_-| \le \omega(x_1) \quad \text{for all } x_1 \ge 0$$

and

$$|\phi'(x_1)| \le \sigma(|x_1|) \quad \text{and} \quad |\Lambda(m_1 - k)(x_1)| \le \tau(|x_1|) \quad \text{for all } x_1 \in \mathbb{R},$$

where $k = \min\{h, 1\}$. Let $r \ge 1$ with

$$\sup_{x_1 \ge r} \omega(x_1) \le \begin{cases} \frac{\alpha}{2} & \text{if } h < 1 \\ \frac{\pi}{2} & \text{if } h > 1. \end{cases}$$

Then for any $R \ge r$ there exists $\tilde{m} \in H^1_{\mathrm{loc}}(\mathbb{R}; \mathbb{S}^1)$ such that

$$\deg(\tilde{m}) = \frac{\ell_+ - \ell_-}{2\pi}, \quad \tilde{m}_1 = k \text{ in } (-\infty, -2R] \cup [2R, \infty), \quad \tilde{m}_1 = m_1 \text{ in } [-R, R],$$

and $|\tilde{m}_1 - k| \le |m_1 - k|$ everywhere, and such that

$$E_h(\tilde{m}) \le E_h(m) + CA \quad \text{if } h < 1$$

and

$$E_h(\tilde{m}) \le E_h(m) + C\left(\frac{1}{R}\int_R^\infty \sigma^2 \, dx_1\right)^{1/2} + CA \quad \text{if } h > 1,$$

where

$$A = B + \left(\int_R^\infty \omega^2 \, dx_1\right)^{1/2} B^{1/2} + \int_R^\infty \omega\tau \, dx_1 \quad \text{and} \quad B = \int_R^\infty \left(\frac{\omega^2}{R^2} + \sigma^2\right) dx_1.$$

Proof. Choose an even function $\eta \in C^{1,1}(\mathbb{R})$ with $\eta(x_1) = 0$ for $|x_1| \ge 1$, $\eta(x_1) = 1$ for $0 \le |x_1| \le \frac{1}{2}$, $\eta(x_1) = (1 - |x_1|)^2$ for $\frac{3}{4} \le |x_1| < 1$, and $\frac{1}{16} \le \eta(x_1) \le 1$ for $\frac{1}{2} < |x_1| < \frac{3}{4}$. Fix $R \ge r$ and set $\tilde{\eta}(x_1) = \eta\left(\frac{x_1}{2R}\right)$ for every $x_1 \in \mathbb{R}$. Now define

$$\tilde{m}_1 = \tilde{\eta} m_1 + (1 - \tilde{\eta})k \quad \text{in } \mathbb{R}.$$

Then clearly $|\tilde{m}_1 - k| = \tilde{\eta}|m_1 - k| \le |m_1 - k|$. It follows in particular that $W(\tilde{m}) \le W(m)$ pointwise in $\mathbb{R}$. Moreover, since the conditions on ω prevent large oscillations of m_1 in $(-\infty, -R] \cup [R, \infty)$, it is clear that there exists $\tilde{m}_2 \colon \mathbb{R} \to [-1, 1]$ such that the map $\tilde{m} = (\tilde{m}_1, \tilde{m}_2)$ belongs to $H^1_{\mathrm{loc}}(\mathbb{R}; \mathbb{S}^1)$ with $\deg(\tilde{m}) = \deg(m) = \frac{\ell_+ - \ell_-}{2\pi}$.

Step 1: estimate $\|\tilde{m}'\|_{L^2(\mathbb{R})}$. We compute

$$\tilde{m}_1' = \tilde{\eta} m_1' + \tilde{\eta}'(m_1 - k) \quad \text{in } \mathbb{R}.$$

We distinguish the cases $h < 1$ and $h > 1$. If $h < 1$, then

$$\frac{1}{C_1} \le 1 - m_1^2 \le C_1(1 - \tilde{m}_1^2) \quad \text{for } |x_1| \ge R,$$

where $C_1 > 0$ is a constant that depends only on α (because of the condition $\sup_{x_1 \ge r} \omega(x_1) \le \frac{\alpha}{2}$). It follows that

$$\frac{(\tilde{m}_1')^2}{1 - \tilde{m}_1^2} \le 2\frac{\tilde{\eta}^2(m_1')^2 + (\tilde{\eta}')^2(m_1 - k)^2}{1 - \tilde{m}_1^2} \le 2C_1\frac{(m_1')^2}{1 - m_1^2} + 2C_1^2(\tilde{\eta}')^2(m_1 - k)^2 \quad \text{for } |x_1| \ge R.$$

Therefore,

$$\int_{-\infty}^\infty \frac{(\tilde{m}_1')^2}{1 - \tilde{m}_1^2} \, dx_1 \le \int_{-R}^R \frac{(m_1')^2}{1 - m_1^2} \, dx_1 + C_2 \int_R^\infty \left(\sigma^2 + \frac{\omega^2}{R^2}\right) dx_1$$

for some constant $C_2 = C_2(\alpha, \eta)$.

If $h > 1$, then $1 + \tilde{m}_1 \ge 1 + m_1 \ge 1$ in $(-\infty, -R] \cup [R, \infty)$ and $1 - \tilde{m}_1 = \tilde{\eta}(1 - m_1)$ in $\mathbb{R}$. Therefore,

$$\frac{(\tilde{m}_1')^2}{1 - \tilde{m}_1^2} \le \tilde{\eta}\frac{(m_1')^2}{1 - m_1} - 2\tilde{\eta}' m_1' + \frac{(\tilde{\eta}')^2}{\tilde{\eta}}(1 - m_1) \quad \text{for } |x_1| \ge R.$$

Clearly, we have

$$\int_{\mathbb{R}\setminus(-R,R)} \tilde{\eta}\frac{(m_1')^2}{1-m_1}\,dx_1 \leq 4\int_R^\infty \sigma^2\,dx_1.$$

By the choice of η, we have $(\eta')^2/\eta \in L^\infty(\mathbb{R})$. Hence there exists a constant $C_3 = C_3(\eta)$, such that

$$\int_{-\infty}^\infty \frac{(\tilde{\eta}')^2}{\tilde{\eta}}(1-m_1)\,dx_1 \leq \frac{C_3}{R^2}\int_{\mathbb{R}\setminus(-R,R)} |1-\cos\phi|\,dx_1 \leq \frac{C_3}{R^2}\int_R^\infty \omega^2\,dx_1.$$

Moreover,

$$-\int_{-\infty}^\infty \tilde{\eta}' m_1'\,dx_1 \leq \frac{C_4}{R}\int_R^{2R}\sigma\,dx_1 \leq C_4\left(\frac{1}{R}\int_R^\infty \sigma^2\,dx_1\right)^{1/2}$$

for a constant $C_4 = C_4(\eta)$. It follows that

$$\int_{-\infty}^\infty \frac{(\tilde{m}_1')^2}{1-\tilde{m}_1^2}\,dx_1 \;\leq\; \int_{-R}^R \frac{(m_1')^2}{1-m_1^2}\,dx_1 \;+\; \int_R^\infty \left(4\sigma^2 + \frac{C_3\omega^2}{R^2}\right)\,dx_1 \;+\; 2C_4\left(\frac{1}{R}\int_R^\infty \sigma^2\,dx_1\right)^{1/2}.$$

Step 2: estimate $\|\tilde{m}_1 - k\|_{\dot{H}^{1/2}(\mathbb{R})}$. We now consider both the cases $h<1$ and $h>1$ together. Note that $\tilde{m}_1 - m_1 = (1-\tilde{\eta})(k-m_1)$, and therefore,

$$\|m_1 - \tilde{m}_1\|_{L^2(\mathbb{R})}^2 \leq 2\int_R^\infty \omega^2\,dx_1.$$

Moreover,

$$\|m_1' - \tilde{m}_1'\|_{L^2(\mathbb{R})}^2 \leq C_5\int_R^\infty \left(\sigma^2 + \frac{\omega^2}{R^2}\right)\,dx_1$$

for a constant $C_5 = C_5(\eta)$. By interpolation, we find that there exists $C_6 = C_6(\eta)$ such that

$$\|m_1 - \tilde{m}_1\|_{\dot{H}^{1/2}(\mathbb{R})}^2 \leq C_6\left(\int_R^\infty \left(\sigma^2 + \frac{\omega^2}{R^2}\right)\,dx_1\right)^{1/2}\left(\int_R^\infty \omega^2\,dx_1\right)^{1/2}.$$

Finally, we consider $v = V(m)$ and $\tilde{v} = V(\tilde{m})$ defined by (8)–(9). We have

$$\int_{\mathbb{R}^2_+} |\nabla\tilde{v}|^2\,dx = \int_{\mathbb{R}^2_+} |\nabla v|^2\,dx + \int_{\mathbb{R}^2_+} |\nabla v - \nabla\tilde{v}|^2\,dx - 2\int_{\mathbb{R}^2_+} \nabla v\cdot(\nabla v - \nabla\tilde{v})\,dx.$$

By the above estimate, we have

$$\int_{\mathbb{R}^2_+} |\nabla v - \nabla\tilde{v}|^2\,dx \overset{(5)}{=} \|m_1 - \tilde{m}_1\|_{\dot{H}^{1/2}(\mathbb{R})}^2 \leq C_6\left(\int_R^\infty \left(\sigma^2 + \frac{\omega^2}{R^2}\right)\,dx_1\right)^{1/2}\left(\int_R^\infty \omega^2\,dx_1\right)^{1/2}.$$

An integration by parts and (12) yield

$$-2\int_{\mathbb{R}^2_+} \nabla v\cdot(\nabla v - \nabla\tilde{v})\,dx = 2\int_{-\infty}^\infty (m_1 - \tilde{m}_1)\Lambda(m_1 - k)\,dx_1 \leq 4\int_R^\infty \omega\tau\,dx_1.$$

Hence

$$\int_{\mathbb{R}^2_+} |\nabla\tilde{v}|^2\,dx_1 \leq \int_{\mathbb{R}^2_+} |\nabla v|^2\,dx + (C_6+4)(A-B),$$

where A and B are defined in the statement of the proposition. Combining these estimates, we obtain the desired inequality for the energy. □

When we apply Proposition 2.1, the following estimate is useful.

Lemma 2.4. *For any $c, C > 0$, there exists a number $R > 0$ such that for any $m \in H^1_{\mathrm{loc}}(\mathbb{R};\mathbb{S}^1)$ and any $x_1 \in \mathbb{R}$, the following holds true. If $E_h(m) \leq C$ and $|m_1(x_1) - k| \geq c$, then $|m_1 - k| \geq c/2$ in $(x_1 - R, x_1 + R)$ and*

$$\int_{x_1-R}^{x_1+R} W(m_1)\,dx_1 \geq \frac{1}{2}\int_{x_1-R}^{x_1+R} (k - m_1(s))^2\,ds \geq \frac{c^2 R}{4}.$$

Proof. Choose $R = \frac{c^2}{16C}$. Then for every $t \in (x_1 - R, x_1 + R)$, we have

$$|m_1(t) - m_1(x_1)|^2 \leq 2R \int_{x_1-R}^{x_1+R} |m_1'(s)|^2\, ds \leq 4RC = \frac{c^2}{4}.$$

The conclusion is now straightforward. $\square$

As a consequence of Proposition 2.1, we have the following localisation result.

Corollary 2.1. *Let $\epsilon > 0$ and $d \in \mathbb{Z} + \{0, \pm\alpha/\pi\}$. Then for any $m \in \mathcal{A}_h(d)$, there exist $\tilde{m} \in \mathcal{A}_h(d)$ and $R > 0$ such that*

$$E_h(\tilde{m}) \leq E_h(m) + \epsilon$$

and $\tilde{m}$ is constant in $(-\infty, -R]$ and in $[R, \infty)$.

Proof. It follows from Lemma 2.4 that $\lim_{x_1 \to \pm\infty} m_1(x_1) = k$. Thus if we choose $\phi \colon \mathbb{R} \to \mathbb{R}$ with $m = (\cos\phi, \sin\phi)$, then Proposition 2.1 applies with $\ell_\pm = \lim_{x_1 \to \pm\infty} \phi(x_1)$ and

$$\begin{aligned}
\omega(x_1) &= |\phi(x_1) - \ell_+| + |\phi(-x_1) - \ell_-|, \\
\sigma(x_1) &= |\phi'(x_1)| + |\phi'(-x_1)|, \\
\tau(x_1) &= |\Lambda(m_1 - k)(x_1)| + |\Lambda(m_1 - k)(-x_1)|,
\end{aligned}$$

provided that $r \geq 1$ is chosen sufficiently large. Since $\omega, \sigma, \tau \in L^2(0, \infty)$, we have

$$\lim_{R \to \infty} \int_R^\infty (\omega^2 + \sigma^2 + \tau^2)\, dx_1 = 0.$$

Thus for a sufficiently large R, the inequalities of Proposition 2.1 lead to the desired conclusion. $\square$

2.4 Monotonicity and subadditivity of the function $\mathcal{E}_h$

In this section, we examine how the number $\mathcal{E}_h(d)$ depends on d. To this end, we construct suitable maps $m \in \mathcal{A}_h(d)$ and estimate their energies.

Proposition 2.2 (Monotonicity)**.** *Suppose that $d_1, d_2 \in \mathbb{Z} + \{0, \pm\alpha/\pi\}$ such that $0 \leq d_1 \leq d_2$. If $h < 1$, suppose that $d_2 - d_1 \neq 1 - \frac{2\alpha}{\pi}$. Then $\mathcal{E}_h(d_1) \leq \mathcal{E}_h(d_2)$.*

Proof. We may assume that $0 < d_1 < d_2$. Suppose that $m \in \mathcal{A}_h(d_2)$. Then there exist $t_1, t_2 \in \mathbb{R} \cup \{\pm\infty\}$ with $t_1 < t_2$ such that[2] $m(t_1) = (\cos\alpha, \pm\sin\alpha)$, $m(t_2) = (\cos\alpha, \pm\sin\alpha)$, and

$$\int_{t_1}^{t_2} m^\perp \cdot m'\, dx_1 = 2\pi d_1.$$

We then define a map $\tilde{m} = (\tilde{m}_1, \tilde{m}_2) \colon \mathbb{R} \to \mathbb{S}^1$ as follows. For $x_1 \in (t_1, t_2)$, we define $\tilde{m}(x_1) = m(x_1)$. For $x_1 \notin (t_1, t_2)$, we define $\tilde{m}_1(x_1) = m_1(x_1)$ and $\tilde{m}_2(x_1) = \pm|m_2(x_1)|$, with the sign locally constant and chosen such that $\tilde{m}_2$ is continuous. Then $\deg(\tilde{m}) = d_1$ and $\tilde{m} \in \mathcal{A}_h(d_1)$. On the other hand, we clearly have $E_h(\tilde{m}) = E_h(m)$. Therefore, we have $\mathcal{E}_h(d_1) \leq E_h(m)$. The desired inequality then follows. $\square$

Proposition 2.3 (Subadditivity)**.** *Suppose that $d_1, d_2, d \in \mathbb{Z} + \{0, \pm\alpha/\pi\}$ with $d = d_1 + d_2$. If $\alpha = \frac{\pi}{3}$ and $d_2 - d_1 \in \mathbb{Z}$, suppose that $d \in \mathbb{Z}$. Then*

$$\mathcal{E}_h(d) \leq \mathcal{E}_h(d_1) + \mathcal{E}_h(d_2).$$

Proof. Choose $m^1 \in \mathcal{A}_h(d_1)$ and $m^2 \in \mathcal{A}_h(d_2)$ and fix $\epsilon > 0$. We want to find $m \in \mathcal{A}_h(d)$ with

$$E_h(m) \leq E_h(m^1) + E_h(m^2) + 3\epsilon. \tag{13}$$

Using Corollary 2.1, choose $R > 0$ and $\tilde{m}^1 \in \mathcal{A}_h(d_1)$ and $\tilde{m}^2 \in \mathcal{A}_h(d_2)$ such that both are constant in $(-\infty, -R]$ and in $[R, \infty)$ and

$$E_h(\tilde{m}^1) \leq E_h(m^1) + \epsilon \quad \text{and} \quad E_h(\tilde{m}^2) \leq E_h(m^2) + \epsilon.$$

There exist $\phi_1, \phi_2 : \mathbb{R} \to \mathbb{R}$ such that $\tilde{m}^1 = (\cos\phi_1, \sin\phi_1)$ and $\tilde{m}^2 = (\cos\phi_2, \sin\phi_2)$. Furthermore, there exist two numbers $\beta_1, \beta_2 \in 2\pi\mathbb{Z} \pm \alpha$ such that $\phi_1 = \beta_1$ in $[R, \infty)$ and $\phi_2 = \beta_2$ in $(-\infty, -R]$. We can assume without loss of generality that $\beta_2 - \beta_1 \in 2\pi\mathbb{Z}$. This can be achieved by either

[2] Here we use the notation $m_1(\pm\infty) = \lim_{x_1 \to \pm\infty} m_1(x_1)$.

- exchanging d_1 and d_2; or
- replacing $\phi_1(x_1)$ by $-\phi_1(-x_1)$ or $\phi_2(x_1)$ by $-\phi_2(-x_1)$; or
- if $\alpha = \frac{\pi}{2}$, replacing ϕ_2 by $\phi_2 + \pi$.

For $r \geq R$, define

$$\psi(x_1) = \begin{cases} \phi_1(x_1 + r) & \text{if } x_1 \leq 0, \\ \phi_2(x_1 - r) - \beta_2 + \beta_1 & \text{if } x_1 > 0. \end{cases}$$

Then obviously we have

$$\int_{-\infty}^{\infty} (\psi')^2 \, dx_1 = \int_{-\infty}^{\infty} (\phi_2')^2 \, dx_1 + \int_{-\infty}^{\infty} (\phi_1')^2 \, dx_1$$

and

$$\int_{-\infty}^{\infty} W(\cos\psi, \sin\psi) \, dx_1 = \int_{-\infty}^{\infty} W(\tilde{m}^1) \, dx_1 + \int_{-\infty}^{\infty} W(\tilde{m}^2) \, dx_1.$$

Let $u^1 = U(\tilde{m}^1)$ and $u^2 = U(\tilde{m}^2)$ be defined by (7). Furthermore, let $w = U(\cos\psi, \sin\psi)$. As (7) determines w uniquely, we deduce that $w(x_1, x_2) = u^1(x_1 + r, x_2) + u^2(x_1 - r, x_2)$. Hence

$$\int_{\mathbb{R}^2_+} |\nabla w|^2 \, dx = \int_{\mathbb{R}^2_+} |\nabla u^1|^2 \, dx + \int_{\mathbb{R}^2_+} |\nabla u^2|^2 \, dx + 2 \int_{\mathbb{R}^2_+} \nabla u^1(x_1 + r, x_2) \cdot \nabla u^2(x_1 - r, x_2) \, dx.$$

Parseval's identity, the dominated convergence theorem, and the Riemann-Lebesgue lemma lead to

$$\int_{\mathbb{R}^2_+} \nabla u^1(x_1 + r, x_2) \cdot \nabla u^2(x_1 - r, x_2) \, dx = \int_0^{\infty} dx_2 \int_{\mathbb{R}} e^{2i\xi r} \mathcal{F}(\nabla u^1)(\xi, x_2) \cdot \overline{\mathcal{F}(\nabla u^2)}(\xi, x_2) \, d\xi \overset{r\to\infty}{\longrightarrow} 0,$$

where we use the fact that $\nabla u^1, \nabla u^2 \in L^2(\mathbb{R}^2_+)$. Hence if r is sufficiently large, the map $m = (\cos\psi, \sin\psi)$ will satisfy (13). By construction, we have $m \in \mathcal{A}_h(d)$, which concludes the proof. $\square$

3 The Euler-Lagrange equation

3.1 Statement and immediate consequences

We now discuss critical points m of the energy E_h. If $m \in \mathcal{A}_h(d)$ is a critical point of E_h, then it is critical relative to $\mathcal{A}_h(d)$ as well, because $\mathcal{A}_h(d)$ is an open set in $\{m \in H^1_{\mathrm{loc}}(\mathbb{R}; \mathbb{S}^1) \colon E_h(m) < \infty\}$ under the strong $\dot{H}^1$-topology. Write $m = (\cos\phi, \sin\phi) \in \mathcal{A}_h(d)$ and let $u = U(m)$ be the function defined by (7). Then the Euler-Lagrange equation is

$$\phi'' = (h - \cos\phi + u')\sin\phi \quad \text{in} \quad \mathbb{R}. \tag{14}$$

We can write the equation in terms of m, noting that $m'' = -(\phi')^2 m + \phi'' m^\perp$. This leads to

$$m'' + |m'|^2 m = (h - m_1 + u') m_2 m^\perp \quad \text{in } \mathbb{R}. \tag{15}$$

Away from $m_1^{-1}(\{\pm 1\})$, we can write the Euler-Lagrange equation in terms of the function

$$f = m_1 - k = \cos\phi - \cos\alpha.$$

Indeed, observing that $1 - m_1^2 = \sin^2\alpha - 2f\cos\alpha - f^2$, we find the equation

$$f'' = -\frac{(f')^2(f + \cos\alpha)}{\sin^2\alpha - 2f\cos\alpha - f^2} + (\sin^2\alpha - 2f\cos\alpha - f^2)(f - \Lambda f - h + k) \quad \text{in } \mathbb{R}\backslash f^{-1}(\{\pm 1 - k\}), \tag{16}$$

where $\Lambda \colon \dot{H}^1(\mathbb{R}) \to L^2(\mathbb{R})$ is the Dirichlet-to-Neumann operator introduced in (11).

The equation admits a regularity theory. In particular, the following can be shown with the arguments of Ignat-Knüpfer [10, Theorem 1.1] (even though they study a slightly different problem).

Proposition 3.1 (Regularity)**.** *If $\phi \in H^1_{\mathrm{loc}}(\mathbb{R})$ with $\cos\phi - k \in \dot{H}^{1/2}(\mathbb{R})$ solves equation* (14)*, then $\phi \in C^\infty(\mathbb{R})$.*

It is an open question whether minimisers of E_h subject to a prescribed winding number (or more general, solutions of (15)) necessarily correspond to a monotone phase ϕ. On the other hand, we can show that a minimiser m will pass through the points $(\pm 1, 0)$ exactly as many times as the winding number requires and in a transversal way.

Lemma 3.1 (Passages through $(\pm 1, 0)$)**.** *Suppose that $m \in \mathcal{A}_h(d)$ minimises E_h in $\mathcal{A}_h(d)$. Then*

$$\begin{cases} |m_1^{-1}(\{\pm 1\})| = 2|d| - 1 & \text{if } h > 1 \text{ and } d \in \mathbb{Z} \setminus \{0\}, \\ |m_1^{-1}(\{\pm 1\})| = 2|d| & \text{if } h < 1 \text{ and } d \in \mathbb{Z}, \\ |m_1^{-1}(\{\pm 1\})| = 2\ell - 1 & \text{if } h < 1 \text{ and } |d| = \ell - 1 + \frac{\alpha}{\pi} \text{ or } |d| = \ell - \frac{\alpha}{\pi} \text{ for some } \ell \in \mathbb{N}. \end{cases}$$

Furthermore, if $a \in \mathbb{R}$ with $m_1(a) = \pm 1$, then $m_2'(a) \neq 0$.

Proof. We may assume that $d \geq 0$. Suppose that $\phi \colon \mathbb{R} \to \mathbb{R}$ is such that $m = (\cos\phi, \sin\phi)$. By Proposition 3.1, we know that ϕ is smooth.

Step 1: prove the second statement. Here we show that $\phi'(a) \neq 0$ if $\phi(a) \in \pi\mathbb{Z}$ for some $a \in \mathbb{R}$. (This will then imply the second statement of the lemma.) To this end, consider the Euler-Lagrange equation in the form (14). Suppose that $\phi(a) = j\pi$ with $j \in \mathbb{Z}$. Then the initial value problem

$$\begin{aligned} \psi'' &= (h - \cos\phi + u')\sin\psi \quad \text{in } \mathbb{R}, \\ \psi(a) &= j\pi, \\ \psi'(a) &= 0, \end{aligned}$$

has the solution $\psi(x_1) = j\pi$. The function ϕ also satisfies the ordinary differential equation and the first initial condition. But since solutions of the initial value problem are unique and ϕ cannot be constant, it follows that ϕ does not satisfy the second initial condition. That is, we have $\phi'(a) \neq 0$. (This kind of argument was also used by Capella-Melcher-Otto in [1].)

Step 2: prove the first statement. Now we show that $\phi(a) < \phi(b)$ for any $a, b \in \mathbb{R}$ with $a < b$, $\phi(a) \in \pi\mathbb{Z}$ and $\phi(b) \in \pi\mathbb{Z}$. (This will imply the first statement of the lemma.) We argue by contradiction here. Suppose that $\phi(a) \geq \phi(b)$. Since

$$\lim_{x_1 \to \infty} \phi(x_1) \geq \lim_{x_1 \to -\infty} \phi(x_1)$$

and there can be no local extrema at a or b by the first part of the proof, it follows that there exist $a', b' \in \mathbb{R}$ with $a' < b'$ such that $\phi(a') = \phi(b') \in \pi\mathbb{Z}$. Now define

$$\tilde{\phi}(x_1) = \begin{cases} \phi(x_1) & \text{if } x_1 \leq a' \text{ or } x_1 \geq b', \\ 2\phi(a') - \phi(x_1) & \text{if } a' < x_1 < b', \end{cases}$$

and $\tilde{m} = (\cos\tilde{\phi}, \sin\tilde{\phi})$. Then $\tilde{m} \in \mathcal{A}_h(d)$ and $\tilde{m}_1 = m_1$. Therefore, we have $E_h(\tilde{m}) = E_h(m)$ and $\tilde{m}$ is another minimiser of E_h in $\mathcal{A}_h(d)$. Proposition 3.1 implies that $\tilde{\phi}$ is smooth. Since we already know that $\phi'(a') \neq 0$, this is impossible. Therefore, we have in fact $\phi(a) < \phi(b)$. $\square$

3.2 Pohozaev identity

Next we prove the Pohozaev identity from Proposition 1.1, which gives equipartition between the exchange and the anisotropy energy for critical points of E_h.

Before we give the rigorous proof, however, we describe the central idea informally. For $t > 0$, let $m^t(x_1) = m(tx_1)$ for every $x_1 \in \mathbb{R}$. We compute $\frac{d}{dt}\big|_{t=1} m^t(x_1) = x_1 m'(x_1)$ and

$$E_h(m^t) = \frac{1}{2}\int_{-\infty}^{\infty} \left(t|m'|^2 + \frac{2}{t}W(m) \right) dx_1 + \frac{1}{2}\int_{-\infty}^{\infty} \left| \left|\tfrac{d}{dx_1}\right|^{\frac{1}{2}} m_1 \right|^2 dx_1,$$

noting that the $\dot{H}^{1/2}$-seminorm is invariant under scaling in $\mathbb{R}$. If m is a critical point of E_h, we expect that

$$0 = \frac{d}{dt}\bigg|_{t=1} E_h(m^t) = \frac{1}{2}\int_{-\infty}^{\infty} \left(|m'|^2 - 2W(m) \right) dx_1.$$

For energy minimisers, the formula from Proposition 1.1 follows in fact immediately. For solutions of the Euler-Lagrange equation, however, we need additional arguments.

Proof of Proposition 1.1. We write $m = (\cos\phi, \sin\phi)$. Let $u = U(m)$ be the function defined in (7). By Proposition 3.1, we know that ϕ is smooth in $\mathbb{R}$.

We now use an argument similar to a proof in our previous paper [11, Lemma 12]. As u is harmonic, we calculate, for every $R > 0$, that

$$\operatorname{div}\left(\frac{1}{2}|\nabla u|^2 x - (x \cdot \nabla u)\nabla u \right) = 0 \quad \text{in } B_R^+ = \left\{ x \in \mathbb{R}^2 \colon |x| < R, x_2 > 0 \right\}.$$

Denote $C_R^+ = \{x \in \partial B_R^+ \colon x_2 > 0\}$ and $\partial_r u = \frac{x}{|x|} \cdot \nabla u$. The Gauss theorem gives

$$\int_{C_R^+} \left(\frac{R}{2} |\nabla u|^2 - R\,(\partial_r u)^2 \right) d\sigma + \int_{-R}^{R} x_1 \frac{\partial u}{\partial x_1} \frac{\partial u}{\partial x_2}\, dx_1 = 0, \quad R > 0.$$

Then (3), (14), and an integration by parts yield

$$\begin{aligned}\int_{-R}^{R} x_1 \frac{\partial u}{\partial x_1} \frac{\partial u}{\partial x_2}\, dx_1 &= \int_{-R}^{R} x_1 \big(\phi'' - \partial_\phi W(\phi)\big)\phi'\, dx_1 \\ &= \frac{1}{2}\Big[x_1\big((\phi')^2 - 2W(\phi)\big)\Big]_{-R}^{R} - \frac{1}{2}\int_{-R}^{R} \big((\phi')^2 - 2W(\phi)\big)\, dx_1.\end{aligned}$$

As $E_h(m) < \infty$ we deduce that the function

$$R \mapsto \left((\phi'(R))^2 + 2W(\phi(R)) + (\phi'(-R))^2 + 2W(\phi(-R)) + \int_{C_R^+} |\nabla u|^2\, d\sigma \right)$$

belongs to $L^1(\mathbb{R}_+)$. Therefore, there exists a sequence $R_k \to \infty$ such that

$$R_k \left((\phi'(R_k))^2 + 2W(\phi(R_k)) + (\phi'(-R_k))^2 + 2W(\phi(-R_k)) + \int_{C_{R_k}^+} |\nabla u|^2\, d\sigma \right) \to 0, \quad k \to \infty.$$

In particular,

$$\Big[x_1\big((\phi')^2 - 2W(\phi)\big)\Big]_{-R_k}^{R_k}, \quad \int_{C_{R_k}^+} \left(\frac{R_k}{2} |\nabla u|^2 - R_k\,(\partial_r u)^2 \right) d\sigma \to 0, \quad k \to \infty.$$

The dominated convergence theorem implies that

$$\frac{1}{2}\int_{-R_k}^{R_k} \big((\phi')^2 - 2W(\phi)\big)\, dx_1 \to \frac{1}{2}\int_{\mathbb{R}} \big((\phi')^2 - 2W(\phi)\big)\, dx_1, \quad k \to \infty.$$

The conclusion is now straightforward. □

3.3 Symmetry

As mentioned previously, if $h < 1$ and $d = \frac{\alpha}{\pi}$ or $d = 1 - \frac{\alpha}{\pi}$, or if $h > 1$ and $d = 1$, then symmetrisation arguments are crucial for the construction of energy minimisers in $\mathcal{A}_h(d)$. Although the same arguments do not work for higher winding numbers, there is still some symmetry.

Definition 3.1. We say that a map $m \colon \mathbb{R} \to \mathbb{S}^1$ is *symmetric* if m_1 is an even function and m_2 is an odd function.

We prove that such symmetry holds true for minimisers of E_h in $\mathcal{A}_h(d)$ with the exception of the case $h < 1$ and $d \in \mathbb{Z}$.

Lemma 3.2 (Symmetry)**.** *Suppose that $d \in \mathbb{Z} \pm \alpha/\pi$ and $m \in \mathcal{A}_h(d)$. Then there exists a symmetric map $m^* \in \mathcal{A}_h(d)$ with $E_h(m^*) \le E_h(m)$. Furthermore, if $m \in \mathcal{A}_h(d)$ is a minimiser of E_h in $\mathcal{A}_h(d)$, then there exists $t_0 \in \mathbb{R}$ such that $m(\,\cdot - t_0)$ is symmetric.*

Proof. Without loss of generality, we may assume that $m_1(0) = 1$ if $d \in 2\mathbb{Z} \pm \alpha/\pi$ and $m_1(0) = -1$ if $d \in 2\mathbb{Z} + 1 \pm \alpha/\pi$ and that

$$\int_{-\infty}^{0} m^\perp \cdot m'\, dx_1 = \int_{0}^{\infty} m^\perp \cdot m'\, dx_1 = \pi d.$$

Define $m^+ = (m_1^+, m_2^+) \in \mathcal{A}_h(d)$ and $m^- = (m_1^-, m_2^-) \in \mathcal{A}_h(d)$ as follows:

$$m_1^+(x_1) = \begin{cases} m_1(x_1) & \text{if } x_1 \ge 0, \\ m_1(-x_1) & \text{if } x_1 < 0, \end{cases} \qquad m_2^+(x_1) = \begin{cases} m_2(x_1) & \text{if } x_1 \ge 0, \\ -m_2(-x_1) & \text{if } x_1 < 0, \end{cases}$$

$$m_1^-(x_1) = \begin{cases} m_1(-x_1) & \text{if } x_1 \ge 0, \\ m_1(x_1) & \text{if } x_1 < 0, \end{cases} \qquad m_2^-(x_1) = \begin{cases} -m_2(-x_1) & \text{if } x_1 \ge 0, \\ m_2(x_1) & \text{if } x_1 < 0. \end{cases}$$

Define $v = V(m)$ and $v^\pm = V(m^\pm)$ as in (8)–(9). Then $\Delta v^+ = 0$ in $\{x_2 > 0\}$ (in particular, v^+ is smooth in $\{x_2 > 0\}$), and by the symmetry of m^+, the function $v^+(\cdot, x_2)$ is even, so that $\frac{\partial v^+}{\partial x_1}(0, x_2) = 0$ for every $x_2 > 0$. Of course, we also have $v^+(x_1, 0) = m_1(x_1) - k$ for $x_1 > 0$. It follows that the restriction of v^+ to $(0,\infty)^2$ is the unique minimiser of the Dirichlet energy in $(0,\infty)^2$ subject to these boundary data on $(0,\infty) \times \{0\}$ and free boundary data on $\{0\} \times (0,\infty)$. In particular,

$$\int_{(0,\infty)^2} |\nabla v^+|^2\, dx \le \int_{(0,\infty)^2} |\nabla v|^2\, dx,$$

with equality if, and only if, $v = v^+$. Similarly,

$$\int_{(-\infty,0)\times(0,\infty)} |\nabla v^-|^2\, dx \le \int_{(-\infty,0)\times(0,\infty)} |\nabla v|^2\, dx,$$

with equality if, and only if, $v = v^-$. Therefore, by the symmetry of $v^\pm$, we have

$$\frac{1}{2}\int_{\mathbb{R}^2_+} \left(|\nabla v^+|^2 + |\nabla v^-|^2\right) dx \le \int_{\mathbb{R}^2_+} |\nabla v|^2\, dx,$$

with equality if, and only if, $v = v_+ = v_-$ (which, in particular, would mean that m_1 is even). It is clear from the construction that

$$\frac{1}{2}\int_{-\infty}^{\infty} \left(|(m^+)'|^2 + |(m^-)'|^2 + 2W(m^+) + 2W(m^-)\right) dx_1 = \int_{-\infty}^{\infty} \left(|m'|^2 + 2W(m)\right) dx_1.$$

Thus we have

$$\frac{1}{2}\left(E_h(m^+) + E_h(m^-)\right) \le E_h(m),$$

with equality if, and only if, m_1 is even. So either m^+ or m^- has the required properties for the first statement.

If m is an energy minimiser, then it follows immediately that m_1 is even. By Lemma 3.1, there exist exactly as many points in $m_1^{-1}(\{\pm 1\})$ as required by the winding number. Therefore, the function m_2 is determined uniquely by m_1 and the winding number, and it follows that m_2 is odd. So m is symmetric. $\square$

3.4 H^2-estimates based on the Euler-Lagrange equation

In this section we use the Euler-Lagrange equation (14) to derive some H^2-estimates for minimisers m of E_h in $\mathcal{A}_h(d)$. Recall that by Lemma 3.1, such a minimiser m passes through the points $(\pm 1, 0)$ a finite number of times, which means, in particular, that $m_2 \neq 0$ on an interval of the form (a, ∞). We prove the following estimate for critical points m of E_h under the assumption that the second component m_2 does not vanish on (a, ∞).

Lemma 3.3. *There exists a universal constant C such that for any solution $\phi \in C^\infty(\mathbb{R})$ of (14) with $u = U(\cos\phi)$, if there exists a number $a \in \mathbb{R}$ such that $\sin\phi \neq 0$ in (a,∞), then*

$$\int_{a+R}^{\infty} \left((\phi'')^2 + (\phi')^2 \sin^2\phi + (\phi')^4(1 + \cot^2\phi)\right) dx_1 + \int_{(a+R,\infty)\times(0,\infty)} |\nabla^2 u|^2\, dx \le \frac{C E_h(m)}{R^2}$$

for any $R > 0$.

Proof. The following arguments rely on ideas from our previous paper [11, Lemma 11]. We first note that

$$\frac{\phi''}{\sin\phi} = h - \cos\phi + u' \quad \text{in } (a,\infty)$$

by (14). Differentiating, we obtain

$$\frac{\phi'''}{\sin\phi} = \frac{\phi''\phi'\cos\phi}{\sin^2\phi} + \phi'\sin\phi + u'',$$

and hence

$$\phi''' = \phi''\phi'\cot\phi + \phi'\sin^2\phi + u''\sin\phi \quad \text{in } (a,\infty). \tag{17}$$

Let $\eta \in C_0^\infty(\mathbb{R}^2)$ with $\eta(x_1, 0) = 0$ for $x_1 \notin (a, \infty)$. Let $v = V(\cos\phi, \sin\phi)$ be defined as in (8)–(9). Then $u''(x_1, 0) = \frac{\partial v'}{\partial x_2}(x_1, 0)$ and $v'(x_1, 0) = -\phi'(x_1)\sin\phi(x_1)$. Multiplying (17) by $\eta^2(\,\cdot\,, 0)\phi'$ and integrating by parts, we obtain

$$\begin{aligned}
\int_a^\infty \eta^2(\phi'')^2\,dx_1 &= -\int_a^\infty \eta^2\big(\underbrace{\phi''(\phi')^2}_{=\frac{1}{3}[(\phi')^3]'}\cot\phi + (\phi')^2\sin^2\phi + u''\underbrace{\phi'\sin\phi}_{=-(\cos\phi)'}\big)\,dx_1 - 2\int_a^\infty \eta\eta'\phi''\phi'\,dx_1 \\
&= -\int_a^\infty \eta^2(\phi')^2\sin^2\phi\,dx_1 - \frac{1}{3}\int_a^\infty \eta^2(\phi')^4(1+\cot^2\phi)\,dx_1 \\
&\quad + 2\int_a^\infty \eta\eta'\left(\frac{1}{3}(\phi')^3\cot\phi - \phi''\phi'\right)\,dx_1 - \int_{\mathbb{R}^2_+} \eta^2|\nabla v'|^2\,dx \\
&\quad - 2\int_{\mathbb{R}^2_+} \eta v'\nabla\eta\cdot\nabla v'\,dx.
\end{aligned}$$

We estimate

$$-2\int_a^\infty \eta\eta'\phi''\phi'\,dx_1 \le \frac{1}{2}\int_a^\infty \eta^2(\phi'')^2\,dx_1 + 2\int_a^\infty (\eta')^2(\phi')^2\,dx_1$$

and

$$\frac{2}{3}\int_a^\infty \eta\eta'(\phi')^3\cot\phi\,dx_1 \le \frac{1}{6}\int_a^\infty \eta^2(\phi')^4\cot^2\phi\,dx_1 + \frac{2}{3}\int_a^\infty (\eta')^2(\phi')^2\,dx_1.$$

Furthermore,

$$-2\int_{\mathbb{R}^2_+} \eta v'\nabla\eta\cdot\nabla v'\,dx \le \frac{1}{2}\int_{\mathbb{R}^2_+} \eta^2|\nabla v'|^2\,dx + 2\int_{\mathbb{R}^2_+} |\nabla\eta|^2|\nabla u|^2\,dx.$$

As u is harmonic and $\nabla u = -\nabla^\perp v$, we have $\frac{\partial^2 u}{\partial x_2^2} = -\frac{\partial^2 u}{\partial x_1^2} = -\frac{\partial v'}{\partial x_2}$. Therefore, the Hessian satisfies the following identity:

$$|\nabla^2 u|^2 = 2|\nabla v'|^2 \quad \text{in } \mathbb{R}^2_+.$$

Hence it follows that

$$\begin{aligned}
\int_a^\infty \eta^2\left((\phi'')^2 + 2(\phi')^2\sin^2\phi + \frac{1}{3}(\phi')^4(1+\cot^2\phi)\right)\,dx_1 + \frac{1}{2}\int_{\mathbb{R}^2_+} \eta^2|\nabla^2 u|^2\,dx \\
\le \frac{16}{3}\int_a^\infty (\eta')^2(\phi')^2\,dx_1 + 4\int_{\mathbb{R}^2_+} |\nabla\eta|^2|\nabla u|^2\,dx.
\end{aligned}$$

A suitable choice of η now gives the desired inequality. □

4 The nonlocal terms

4.1 Some estimates in $\dot{H}^{1/2}$

Here we derive some inequalities that we will use to estimate the stray field energy $\frac{1}{2}\int_{\mathbb{R}^2_+} |\nabla u|^2\,dx$ appearing in $\mathcal{E}_h$. This part of the energy is the most difficult to control and is chiefly responsible for the interesting pattern of existence and nonexistence of minimisers described in Sect. 1.

As we have seen, we can write

$$\int_{\mathbb{R}^2_+} |\nabla u|^2\,dx = \|m_1 - k\|^2_{\dot{H}^{1/2}(\mathbb{R})}$$

if $u \in \dot{H}^1(\mathbb{R}^2_+)$ is the unique solution of (2)–(3). Therefore, the subsequent analysis is also about the space $\dot{H}^{1/2}(\mathbb{R})$ and its inner product $\langle\cdot,\cdot\rangle_{\dot{H}^{1/2}(\mathbb{R})}$, which can be expressed either through harmonic extensions to $\mathbb{R}^2_+$ or by [13, Theorem 7.12]:

$$\begin{aligned}
\langle f, g\rangle_{\dot{H}^{1/2}(\mathbb{R})} &= -\int_{\mathbb{R}} \Lambda f\, g\,dx_1 = \frac{1}{2\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty \frac{(f(s)-f(t))(g(s)-g(t))}{(s-t)^2}\,ds\,dt \qquad (18) \\
&\overset{(10),(12)}{=} \int_{\mathbb{R}^2_+} \nabla V(f)\cdot\nabla V(g)\,dx = \int_{\mathbb{R}^2_+} \nabla U(f)\cdot\nabla U(g)\,dx,
\end{aligned}$$

for $f, g \in \dot{H}^{1/2}(\mathbb{R})$. From this formula we obtain some inequalities in particular if f and g have disjoint or almost disjoint supports.

Lemma 4.1 (Repulsion between positive and negative parts)**.** *Let $f \in \dot{H}^{1/2}(\mathbb{R})$ and define $f_+ = \max\{f, 0\} \geq 0$ and $f_- = \min\{f, 0\} \leq 0$. Then*

$$\|f\|^2_{\dot{H}^{1/2}(\mathbb{R})} \geq \|f_+\|^2_{\dot{H}^{1/2}(\mathbb{R})} + \|f_-\|^2_{\dot{H}^{1/2}(\mathbb{R})},$$

with equality if, and only if, f does not change sign (i.e., either $f_+ = 0$ or $f_- = 0$).

Proof. By the bilinearity, this statement is equivalent to

$$\langle f_+, f_- \rangle_{\dot{H}^{1/2}(\mathbb{R})} \geq 0,$$

with equality if, and only if, either $f_+ = 0$ or $f_- = 0$. Using (18) and the fact that $f_+ f_- = 0$ in $\mathbb{R}$, we obtain

$$\langle f_+, f_- \rangle_{\dot{H}^{1/2}(\mathbb{R})} = -\frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f_+(s) f_-(t)}{(s-t)^2} \, ds \, dt.$$

It is clear that the right-hand side has the required properties. □

The following inequalities are based on similar ideas.

Lemma 4.2. *Suppose that $f, g \in L^2(\mathbb{R}) \cap \dot{H}^{1/2}(\mathbb{R})$ and there exist $a \in \mathbb{R}$ and $R > 0$ such that $\operatorname{supp} f \subset (-\infty, a - R]$ and $\operatorname{supp} g \subset [a + R, \infty)$. Then*

$$\left| \langle f, g \rangle_{\dot{H}^{1/2}(\mathbb{R})} \right| \leq \frac{\|f\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})}}{2\pi R \sqrt{6}}.$$

Proof. We may assume that $a = 0$. We have

$$\begin{aligned}
\left| \langle f, g \rangle_{\dot{H}^{1/2}(\mathbb{R})} \right| &= \frac{1}{2\pi} \left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{(f(s) - f(t))(g(s) - g(t))}{(s-t)^2} \, ds \, dt \right| \\
&\leq \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|f(s)||g(t)|}{(s-t)^2} \, ds \, dt.
\end{aligned}$$

For any $t \geq R$,

$$\int_{-\infty}^{\infty} \frac{|f(s)|}{(s-t)^2} \, ds \leq \|f\|_{L^2(\mathbb{R})} \left(\int_{-\infty}^{-R} \frac{ds}{(s-t)^4} \right)^{1/2} = \frac{\|f\|_{L^2(\mathbb{R})}}{\sqrt{3(t+R)^3}}.$$

Thus

$$\begin{aligned}
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|f(s)||g(t)|}{(s-t)^2} \, ds \, dt &\leq \|f\|_{L^2(\mathbb{R})} \int_R^{\infty} \frac{|g(t)|}{\sqrt{3(t+R)^3}} \, dt \\
&\leq \|f\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})} \left(\int_R^{\infty} \frac{dt}{3(t+R)^3} \right)^{1/2} = \frac{\|f\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})}}{2R\sqrt{6}}.
\end{aligned}$$

The claim now follows. □

Lemma 4.3. *Suppose that $f, g \in \dot{H}^{1/2}(\mathbb{R})$ are nonnegative functions and $R > 0$ with $\operatorname{supp} f \subset [-2R, -R]$ and $\operatorname{supp} g \subset [R, 2R]$. Then*

$$-\frac{1}{4\pi R^2} \|f\|_{L^1(\mathbb{R})} \|g\|_{L^1(\mathbb{R})} \leq \langle f, g \rangle_{\dot{H}^{1/2}(\mathbb{R})} \leq -\frac{1}{16\pi R^2} \|f\|_{L^1(\mathbb{R})} \|g\|_{L^1(\mathbb{R})}.$$

Proof. Again we have

$$\langle f, g \rangle_{\dot{H}^{1/2}(\mathbb{R})} = -\frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(s) g(t)}{(s-t)^2} \, ds \, dt.$$

But $t - s \leq 4R$ for $t \in \operatorname{supp} g$ and $s \in \operatorname{supp} f$. Hence

$$\langle f, g \rangle_{\dot{H}^{1/2}(\mathbb{R})} \leq -\frac{1}{16\pi R^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(s) g(t) \, ds \, dt = -\frac{1}{16\pi R^2} \|f\|_{L^1(\mathbb{R})} \|g\|_{L^1(\mathbb{R})}.$$

The other inequality follows similarly. □

4.2 Pointwise estimates for the Dirichlet-to-Neumann operator

When analysing the Euler-Lagrange equation for minimisers of E_h, we need to control in particular the term involving the non-local Dirichlet-to-Neumann operator Λ defined by (11) (written as u' in (14)). In this section we derive some pointwise estimates that will help to achieve this.

Lemma 4.4. *For any $f \in H^2(\mathbb{R})$, any $a \in \mathbb{R}$ and any $R \geq 1$,*

$$|\Lambda f(a+R)| \leq \frac{21}{R}\|f\|_{L^2(\mathbb{R})} + 9\|f'\|_{L^2(a,\infty)} + \|f''\|_{L^2(a,\infty)}.$$

Proof. We may assume that $a = 0$. Let

$$\chi(x_1) = \begin{cases} 0 & \text{if } x_1 \leq 0, \\ 8x_1^2/R^2 & \text{if } 0 < x_1 \leq R/4, \\ 1 - 2(1 - 2x_1/R)^2 & \text{if } R/4 < x_1 \leq R/2, \\ 1 & \text{if } x_1 > R/2. \end{cases}$$

Then $\chi \in C^{1,1}(\mathbb{R})$ with $|\chi'| \leq 4/R$ and $|\chi''| \leq 16/R^2$. We split Λ into two operators: for $f \in H^2(\mathbb{R})$, let

$$\Lambda_+ f = \Lambda(\chi f) \quad \text{and} \quad \Lambda_- f = \Lambda((1-\chi)f).$$

Then it follows from Plancherel's theorem that

$$\|\Lambda_+ f\|_{L^2(\mathbb{R})} = \|(\chi f)'\|_{L^2(\mathbb{R})} \leq \frac{4}{R}\|f\|_{L^2(0,\infty)} + \|f'\|_{L^2(0,\infty)}.$$

Moreover,

$$\|(\Lambda_+ f)'\|_{L^2(\mathbb{R})} = \|(\chi f)''\|_{L^2(\mathbb{R})} \leq \frac{16}{R^2}\|f\|_{L^2(0,\infty)} + \frac{8}{R}\|f'\|_{L^2(0,\infty)} + \|f''\|_{L^2(0,\infty)}.$$

Both inequalities combined imply that

$$\begin{aligned} |\Lambda_+ f(R)| &\leq \|\Lambda_+ f\|_{L^1(R,R+1)} + \|(\Lambda_+ f)'\|_{L^1(R,R+1)} \\ &\leq \frac{20}{R}\|f\|_{L^2(0,\infty)} + 9\|f'\|_{L^2(0,\infty)} + \|f''\|_{L^2(0,\infty)}. \end{aligned}$$

For $\Lambda_- f$, we have

$$\Lambda_- f(R) = \frac{1}{\pi}\int_{-\infty}^{R/2} \frac{(1-\chi(t))f(t)}{(t-R)^2}\, dt$$

by (11). Hence

$$|\Lambda_- f(R)| \leq \frac{1}{\pi}\left(\int_{R/2}^{\infty} \frac{dt}{t^4}\right)^{1/2} \|f\|_{L^2(\mathbb{R})} \leq R^{-3/2}\|f\|_{L^2(\mathbb{R})}.$$

Combining these estimates, we finally obtain the desired inequality. ☐

Lemma 4.5. *There exists a universal constant C with the following property. Suppose that $\phi \in C^\infty(\mathbb{R})$ is a solution of (14) and there exists a number $a \in \mathbb{R}$ such that $\sin\phi \neq 0$ in (a,∞). Then for $x_1 > a+1$,*

$$|\Lambda(\cos\phi - k)(x_1)| \leq \frac{C}{x_1 - a}\sqrt{\frac{E_h(\cos\phi, \sin\phi)}{\min\{1, h-1\}}} \quad \text{if } h > 1$$

and

$$|\Lambda(\cos\phi - k)(x_1)| \leq \frac{C}{x_1 - a}\sqrt{E_h(\cos\phi, \sin\phi)} \quad \text{if } h < 1.$$

Proof. Set $m = (\cos\phi, \sin\phi)$ and $f = \cos\phi - k$. Then by Lemma 3.3, we have a universal constant C_1 such that for every $R > 0$:

$$\|f''\|_{L^2(a+R,\infty)} \leq \|\phi'\|_{L^4(a+R,\infty)}^2 + \|\phi''\|_{L^2(a+R,\infty)} \leq \frac{C_1}{R}\sqrt{E_h(m)}$$

and

$$\|f'\|_{L^2(a+R,\infty)} = \|\phi'\sin\phi\|_{L^2(a+R,\infty)} \leq \frac{C_1}{R}\sqrt{E_h(m)}.$$

If $h > 1$, then $W(m) \geq (h-1)|f|$, so that

$$\|f\|_{L^2(\mathbb{R})} \leq \sqrt{2\|f\|_{L^1(\mathbb{R})}} \leq \sqrt{2\frac{E_h(m)}{h-1}}.$$

If $h < 1$, then

$$\|f\|_{L^2(\mathbb{R})} \leq \sqrt{2E_h(m)}.$$

Hence the claim follows from Lemma 4.4. $\square$

The following is another useful estimate based on a cut-off argument as in Lemma 4.4.

Proposition 4.1. *Let $p \in (1,2)$ and $q \in [1,\infty)$. Then there exists a constant $C = C(p,q) > 0$ such that the following holds true. Suppose that $f \in \dot{H}^{1/2}(\mathbb{R}) \cap H^2_{\text{loc}}(\mathbb{R}) \cap L^q(\mathbb{R})$ and $a \in \mathbb{R}$. Then for any $R > 0$,*

$$|\Lambda f(a)| \leq \frac{C\,(1+|\log R|)}{R^{1+1/p}}\left(R^2\|f''\|_{L^p(a-R,a+R)} + \|f\|_{L^p(a-R,a+R)}\right) + \frac{C}{R^{1+1/q}}\|f\|_{L^q(\mathbb{R})}.$$

For the proof, we need the following inequalities.

Lemma 4.6. *For every $p \in (1,\infty)$ and every $R > 0$,*

$$\int_0^R |\log t|^p\,dt \leq pR|\log R|^p + p^pR.$$

Proof. An integration by parts and Hölder's and Young's inequalities imply

$$\begin{aligned}\int_0^R |\log t|^p\,dt &= R|\log R|^p - p\int_0^R |\log t|^{p-2}\log t\,dt \\ &\leq R|\log R|^p + pR^{1/p}\left(\int_0^R |\log t|^p\,dt\right)^{\frac{p-1}{p}} \\ &\leq R|\log R|^p + p^{p-1}R + \frac{p-1}{p}\int_0^R |\log t|^p\,dt,\end{aligned}$$

and the claim follows. $\square$

Lemma 4.7. *Let $I \subset \mathbb{R}$ be a bounded, open interval. Suppose that $p \in (1,2)$ and $f \in W^{2,p}(I)$. Then for any $\chi \in C_0^{1,1}(I) \setminus \{0\}$,*

$$\|\chi' f'\|_{L^p(I)} \leq \frac{\|\chi'\|^2_{L^\infty(I)}\|f''\|_{L^p(I)}}{2\|\chi''\|_{L^\infty(I)}} + \frac{2}{p-1}\|\chi''\|_{L^\infty(I)}\|f\|_{L^p(I)}.$$

Proof. For $\epsilon > 0$, set $f_\epsilon = \sqrt{f^2+\epsilon^2}$ and note that $f'_\epsilon = ff'/f_\epsilon$ and $f''_\epsilon = ff''/f_\epsilon + \epsilon^2(f')^2/f^3_\epsilon \geq ff''/f_\epsilon$. Hence using Hölder's inequality, an integration by parts, and Hölder's inequality again, we find that

$$\begin{aligned}\int_I |\chi'|^p|f'_\epsilon|^p\,dx_1 &\leq \left(\int_I (\chi')^2(f'_\epsilon)^2f_\epsilon^{p-2}\,dx_1\right)^{p/2}\left(\int_I f_\epsilon^p\,dx_1\right)^{1-p/2} \\ &= \left(\frac{1}{1-p}\int_I \left((\chi')^2f''_\epsilon f_\epsilon^{p-1} + 2\chi'\chi''f'_\epsilon f_\epsilon^{p-1}\right)\,dx_1\right)^{p/2}\left(\int_I f_\epsilon^p\,dx_1\right)^{1-p/2} \\ &\leq \left(\frac{1}{1-p}\int_I \left((\chi')^2f''ff_\epsilon^{p-2} + 2\chi'\chi''f'_\epsilon f_\epsilon^{p-1}\right)\,dx_1\right)^{p/2}\left(\int_I f_\epsilon^p\,dx_1\right)^{1-p/2} \\ &\leq \left(\frac{1}{p-1}\|\chi'\|^2_{L^\infty(I)}\|f''\|_{L^p(I)} + \frac{2}{p-1}\|\chi''\|_{L^\infty(I)}\|\chi'f'_\epsilon\|_{L^p(I)}\right)^{p/2}\|f_\epsilon\|^{p/2}_{L^p(I)}.\end{aligned}$$

Now Young's inequality yields

$$\begin{aligned}\|\chi'f'_\epsilon\|_{L^p(I)} &\leq \left(\|\chi'f'_\epsilon\|_{L^p(I)} + \frac{\|\chi'\|^2_{L^\infty(I)}\|f''\|_{L^p(I)}}{2\|\chi''\|_{L^\infty(I)}}\right)^{1/2}\left(\frac{2}{p-1}\|\chi''\|_{L^\infty(I)}\|f_\epsilon\|_{L^p(I)}\right)^{1/2} \\ &\leq \frac{1}{2}\|\chi'f'_\epsilon\|_{L^p(I)} + \frac{\|\chi'\|^2_{L^\infty(I)}\|f''\|_{L^p(I)}}{4\|\chi''\|_{L^\infty(I)}} + \frac{1}{p-1}\|\chi''\|_{L^\infty(I)}\|f_\epsilon\|_{L^p(I)}.\end{aligned}$$

We conclude that

$$\|\chi' f_\epsilon'\|_{L^p(I)} \le \frac{\|\chi'\|^2_{L^\infty(I)}\|f''\|_{L^p(I)}}{2\|\chi''\|_{L^\infty(I)}} + \frac{2}{p-1}\|\chi''\|_{L^\infty(I)}\|f_\epsilon\|_{L^p(I)}.$$

The claim now follows from Lebesgue's dominated convergence theorem. □

Proof of Proposition 4.1. We may assume without loss of generality that $a = 0$. Let $v \in \dot{H}^1(\mathbb{R}^2_+)$ be the harmonic extension of f to the half-plane, i.e., $v = V(f)$ as defined in (10). By the Poisson formula, we have

$$v(x_1, x_2) = \frac{x_2}{\pi}\int_{-\infty}^{\infty} \frac{f(t)}{(t-x_1)^2 + x_2^2}\, dt.$$

As in the proof of Lemma 4.4, we choose a cut-off function $\chi \in C_0^{1,1}(-R, R)$ with $0 \le \chi \le 1$ and with $\chi \equiv 1$ in $(-R/2, R/2)$, such that

$$|\chi'| \le 4/R \quad \text{and} \quad |\chi''| \le 16/R^2. \tag{19}$$

We decompose

$$v = v_0 + v_1, \quad v_0 = V(\chi f), \quad v_1 = V\big((1-\chi)f\big);$$

that is,

$$v_0(x_1,x_2) = \frac{x_2}{\pi}\int_{-\infty}^{\infty}\frac{f(t)\chi(t)}{(t-x_1)^2+x_2^2}\,dt \quad \text{and} \quad v_1(x_1,x_2) = \frac{x_2}{\pi}\int_{-\infty}^{\infty}\frac{f(t)(1-\chi(t))}{(t-x_1)^2+x_2^2}\,dt.$$

By (12), we have

$$|\Lambda f(0)| = \left|\frac{\partial v}{\partial x_2}(0,0)\right| \le \left|\frac{\partial v_0}{\partial x_2}(0,0)\right| + \left|\frac{\partial v_1}{\partial x_2}(0,0)\right|.$$

Step 1: estimate for $\frac{\partial v_1}{\partial x_2}(0,0)$. For any $q > 1$, we have the estimate

$$\begin{aligned}\left|\frac{\partial v_1}{\partial x_2}(0,0)\right| &\le \frac{1}{\pi}\int_{\mathbb{R}\setminus(-R/2,R/2)}\frac{|f(t)|}{t^2}\,dt \\ &\le \frac{1}{\pi}\left(2\int_{R/2}^{\infty}\frac{dt}{t^{2q/(q-1)}}\right)^{\frac{q-1}{q}}\|f\|_{L^q(\mathbb{R})} = \frac{1}{\pi}\left(\frac{2q-2}{q+1}\right)^{\frac{q-1}{q}}\left(\frac{R}{2}\right)^{-\frac{q+1}{q}}\|f\|_{L^q(\mathbb{R})}.\end{aligned}$$

A similar inequality also holds if $q = 1$.

Step 2: estimate for $\frac{\partial v_0}{\partial x_2}(0,0)$. We write $g = \chi f \in H^2(\mathbb{R})$ with $\operatorname{supp} g \subset [-R, R]$. For v_0, we then perform the change of variables $t = x_2 s + x_1$ and obtain

$$v_0(x_1,x_2) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{g(x_2 s + x_1)}{s^2+1}\,ds.$$

Hence

$$\frac{\partial v_0}{\partial x_2}(x_1,x_2) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{s g'(x_2 s + x_1)}{s^2+1}\,ds.$$

As

$$\frac{d}{ds}\left(\frac{1}{2}\log\left(x_2^2 s^2 + x_2^2\right)\right) = \frac{s}{s^2+1},$$

an integration by parts yields

$$\begin{aligned}\frac{\partial v_0}{\partial x_2}(x_1,x_2) &= -\frac{x_2}{2\pi}\int_{-\infty}^{\infty} g''(x_2 s + x_1)\log(x_2^2 s^2 + x_2^2)\,ds \\ &= -\frac{1}{2\pi}\int_{-\infty}^{\infty} g''(t)\log((t-x_1)^2 + x_2^2)\,dt.\end{aligned}$$

In particular,

$$\frac{\partial v_0}{\partial x_2}(0,0) = -\frac{1}{\pi}\int_{-\infty}^{\infty} g''(t)\log|t|\,dt,$$

which implies, for $p \in (1,2)$, that

$$\left|\frac{\partial v_0}{\partial x_2}(0,0)\right| \le \frac{1}{\pi}\left(2\int_0^R |\log t|^{p/(p-1)}\,dt\right)^{\frac{p-1}{p}}\|g''\|_{L^p(\mathbb{R})}.$$

As a consequence of this and Lemma 4.6, we obtain a constant $C_1 = C_1(p)$ such that

$$\left|\frac{\partial v_0}{\partial x_2}(0,0)\right| \le C_1 \left(1 + |\log R|\right) R^{(p-1)/p} \|g''\|_{L^p(\mathbb{R})}.$$

It remains to estimate the L^p-norm of g''. To this end, we observe that $g'' = \chi f'' + 2\chi' f' + \chi'' f$. Hence

$$\|g''\|_{L^p(\mathbb{R})} \le \|f''\|_{L^p(-R,R)} + 2\|\chi' f'\|_{L^p(\mathbb{R})} + \|\chi''\|_{L^\infty(\mathbb{R})}\|f\|_{L^p(-R,R)}.$$

Lemma 4.7 provides an estimate for the second term. Using (19), we then see that there exists a constant $C_2 = C_2(p)$ satisfying

$$\|g''\|_{L^p(\mathbb{R})} \le 2\|f''\|_{L^p(-R,R)} + \frac{C_2}{R^2}\|f\|_{L^p(-R,R)}.$$

Now it suffices to combine the above inequalities. $\square$

5 Analysis of the Euler-Lagrange equation

We now analyse the Euler-Lagrange equation for minimisers $m = (\cos\phi, \sin\phi)$ of E_h in $\mathcal{A}_h(d)$ for a given $d \in \mathbb{N}$ in the case $h > 1$ and for $d = \alpha/\pi$ or $d = 1 - \alpha/\pi$ in the case $h < 1$. Of particular interest is the rate of decay of m_1 near $\pm\infty$.

5.1 Exponential decay for $h > 1$

We proceed to establish exponential decay of minimisers ϕ and its derivatives. To this end, we first prove the following lemmas.

Lemma 5.1. *Let $h > 1$ and $a > 0$, and let $\phi\colon \mathbb{R} \to \mathbb{R}$ be a smooth function such that $1 - \cos\phi \in \dot{H}^{1/2}(\mathbb{R})$ and*

$$\begin{cases} \phi \text{ is solution of (14) in } (a,\infty), \\ 0 < \phi < \pi \quad \text{and} \quad |\Lambda(1-\cos\phi)| \le \frac{h-1}{2} \quad \text{in } (a,\infty). \end{cases}$$

Then $\phi' \le 0$ in $[a,\infty)$.

Proof. Suppose, by way of contradiction, that there exists $b \ge a$ with $\phi'(b) > 0$. Then there exists $c > b$ such that $\phi' > 0$ in $[b,c)$ and $\phi'(c) = \phi'(b)/2$. As $\sin\phi > 0$ and $|\Lambda(1-\cos\phi)| \le \frac{h-1}{2}$ in (a,∞), equation (14) implies

$$\phi'' \ge \frac{1}{2}(h - \cos\phi)\sin\phi \quad \text{in } (a,\infty). \tag{20}$$

Hence

$$\frac{d}{dx_1}(\phi'(x_1))^2 \ge [(h-\cos\phi)\sin\phi]\,\phi' > 0 \quad \text{in } (b,c).$$

It follows that $\phi'(c) > \phi'(b) > 0$, in contradiction to the choice of c. $\square$

Proposition 5.1. *Let $h > 1$. Then there exists a constant $c > 0$ with the following property. Let $a > 0$ and let $\phi\colon \mathbb{R} \to \mathbb{R}$ be a smooth function such that $1 - \cos\phi \in \dot{H}^{1/2}(\mathbb{R})$ and*

$$\begin{cases} \phi \text{ is solution of (14) in } (a,\infty), \\ 0 < \phi \le 1 \quad \text{and} \quad |\Lambda(1-\cos\phi)| \le \frac{h-1}{2} \quad \text{in } (a,\infty), \\ \lim_{x_1\to\infty}\phi(x_1) = 0. \end{cases}$$

Then

$$\phi(x_1) \le e^{c(a-x_1)} \quad \text{for all } x_1 \ge a.$$

Remark 5.1. It will not be necessary to know the value of c explicitly, but we will prove the inequality for $c = \gamma\sqrt{h-1}$, where γ is the constant introduced in Lemma 2.3.

Proof. Under the hypotheses of the lemma, equation (14) gives rise to the inequality (20) in (a,∞) again. As $\phi' \le 0$ in $[a,\infty)$ by Lemma 5.1, this implies that $\limsup_{x_1\to\infty}\phi'(x_1) \le 0$ and

$$\frac{d}{dx_1}\left((\phi'(x_1))^2 - \frac{1}{2}(h - \cos\phi(x_1))^2\right) \le 0 \quad \text{in } (a,\infty).$$

As $\lim_{x_1 \to \infty} \phi(x_1) = 0$, we deduce $\limsup_{x_1 \to \infty} \phi'(x_1) = 0$ and $\lim_{x_1 \to \infty} \cos \phi = 1$, so it follows that

$$(\phi'(x_1))^2 \geq \frac{1}{2} \left(\cos^2 \phi(x_1) - 2h \cos \phi(x_1) + 2h - 1 \right) = W(\cos \phi(x_1), \sin \phi(x_1)) \quad \text{for all } x_1 \geq a.$$

Therefore,

$$\phi'(x_1) \leq -\sqrt{W(\cos \phi(x_1), \sin \phi(x_1))} \quad \text{for all } x_1 \geq a.$$

Since $W(\cos \phi, \sin \phi) \geq c^2 \phi^2$ for $c = \gamma \sqrt{h-1}$ by Lemma 2.3, we conclude that $\phi' \leq -c\phi$ in $[a, \infty)$, from which we finally obtain the desired inequality. □

For minimisers in $\mathcal{A}_h(d)$ with $d \in \mathbb{Z}$, we can now prove exponential decay at $\pm\infty$. For convenience, we consider *negative* winding numbers in the statement of the next result, but of course we immediately obtain a statement for positive winding numbers as well.

Theorem 5.1 (Exponential decay for $h > 1$)**.** *Let $h > 1$, $d \in \mathbb{N}$, $\beta < 2$, and let $m = (\cos \phi, \sin \phi) \in \mathcal{A}_h(-d)$ be a minimiser of E_h in $\mathcal{A}_h(-d)$ such that*

$$\lim_{x_1 \to \infty} \phi(x_1) = 0.$$

Then there exist $a \in \mathbb{R}$ and $c, C > 0$ such that for all $x_1 \geq a$: $\phi'(x_1) \leq 0$ and

$$\max\{|\phi(x_1)|, |\phi'(x_1)|, |\phi''(x_1)|\} \leq e^{c(a-x_1)} \tag{21}$$

and

$$|\Lambda(m_1 - 1)(x_1)| \leq \frac{C}{(x_1 - a)^\beta}.$$

Proof. By Proposition 3.1, we know that ϕ is smooth. By the hypothesis and Lemma 3.1, there exists $a' \geq 1$ such that

$$0 < \sin \phi \leq \phi \leq 1 \quad \text{in } [a', \infty).$$

(The fact that the degree of m is $-d < 0$ is essential for the positive sign of $m_2 = \sin \phi$ near $+\infty$.) Moreover, by Lemma 4.5, we may assume that

$$|\Lambda(1 - \cos \phi)| \leq \frac{h-1}{2} \quad \text{in } [a', \infty)$$

as well. Hence, Lemma 5.1 implies that ϕ is monotone in $[a', \infty)$; also, we may apply Proposition 5.1 and we obtain a constant $c > 0$ such that

$$\phi(x_1) \leq e^{c(a' - x_1)} \quad \text{for } x_1 \geq a'.$$

Using equation (14), we then obtain

$$|\phi''(x_1)| \leq \frac{3h-1}{2} \phi(x_1) \leq \frac{3h-1}{2} e^{c(a'-x_1)} \quad \text{for } x_1 \geq a'.$$

If $a'' \geq a'$ is chosen sufficiently large, then it follows that $|\phi''(x_1)| \leq e^{c(a''-x_1)}$ for $x_1 \geq a''$. Since $\liminf_{x_1 \to \infty} |\phi'(x_1)| = 0$ (because $\phi(x_1) \to 0$ as $x_1 \to \infty$), this implies

$$|\phi'(x_1)| \leq \int_{x_1}^{\infty} |\phi''(t)| \, dt \leq \frac{1}{c} e^{c(a''-x_1)} \quad \text{for } x_1 \geq a''.$$

Choosing a sufficiently large, we obtain inequality (21).

It remains to establish the decay of $\Lambda(m_1 - 1)$ at ∞. Lemma 4.5 already gives the decay $1/x_1$ as $x_1 \to \infty$. In order to improve it, we may assume without loss of generality that inequalities similar to (21) hold for $2\pi d - \phi(x_1)$ and for the derivatives $\phi'(x_1)$ and $\phi''(x_1)$ when $x_1 \leq -a''$ (because the behaviour of ϕ as $x_1 \to -\infty$ is similar, albeit with limit $2\pi d$). Fix $p \in (1, 2)$ such that $\beta < 1 + 1/p$. Then it follows immediately that

$$\| \cos \phi - 1 \|_{L^p(\mathbb{R})} \leq C_1$$

for a constant C_1 that depend only on p, c and a''. Moreover, for every $x_1 \geq 2a''$ and $R = \frac{x_1 - a''}{2}$, we have the inequality

$$\int_{x_1 - R}^{x_1 + R} \left(|\phi''(t)|^p + |\phi'(t)|^{2p} \right) \, dt \leq C_2 e^{cp(a''-x_1)/2},$$

where $C_2 = C_2(p, c, a'')$. We apply Proposition 4.1 for $f = 1 - \cos\phi$ and $q = p$. Since $|f''| \le |\phi''| + |\phi'|^2$, then there exists a constant C_3 with

$$|\Lambda(1-\cos\phi)(x_1)| \le \frac{C_3(1 + |\log(x_1 - a'')|)}{(x_1 - a'')^{1+1/p}}[1 + (x_1 - a'')^2 e^{c(a''-x_1)/2}]$$

for all $x_1 \ge 2a''$. If we choose $a \ge 2a''$ large enough, then the desired inequality follows for all $x_1 \ge a$. □

5.2 The linearised equation for $h < 1$

When $h = \cos\alpha \in [0, 1)$ with $\alpha \in (0, \frac{\pi}{2}]$, we will not obtain exponential decay of the minimising profile, because the contribution of the non-local differential operator in (14) is no longer dominated by the local terms. Our analysis here is motivated by the analysis of Chermisi-Muratov [2] for the winding numbers $\frac{\alpha}{\pi}$ and $1 - \frac{\alpha}{\pi}$. An important tool is the fundamental solution of the linearisation of (14) about the trivial solution $\phi_0 = \alpha$, which is calculated in the aforementioned work. The paper also gives estimates for the fundamental solution, which we improve somewhat here.

We consider the differential operator L, given by[3]

$$L\psi = -\psi'' + \psi - \sin\alpha\,\Lambda\psi. \tag{22}$$

The fundamental solution G_α for the equation $L\psi = 0$ (satisfying $LG_\alpha = \delta_0$, where δ_0 is the Dirac measure at 0) is computed, using the Fourier transform and contour integration, by Chermisi–Muratov [2, Lemma A.1]. It is

$$G_\alpha(x_1) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{i\xi x_1}\,d\xi}{\xi^2 + 1 + |\xi|\sin\alpha} = \frac{\sin\alpha}{\pi}\int_0^\infty \frac{te^{-t|x_1|}}{t^2\sin^2\alpha + (t^2-1)^2}\,dt \quad \text{for all } x_1 \in \mathbb{R}. \tag{23}$$

That is, for a solution $g \in H^2(\mathbb{R})$ of the equation $Lg = f$ with $f \in L^2(\mathbb{R})$, we have

$$g = G_\alpha * f.$$

Lemma 5.2. *There exists a constant $C > 0$ such that for any $\alpha \in (0, \frac{\pi}{2}]$, the fundamental solution G_α of the operator L defined in (22) satisfies, for all $x_1 \neq 0$, the inequalities*

$$0 \le G_\alpha(x_1) \le \frac{C\sin\alpha}{1 + x_1^2} + Ce^{-|x_1|/2}$$

and

$$0 \le -\frac{x_1}{|x_1|}G'_\alpha(x_1) \le \frac{C\sin\alpha}{1 + |x_1|^3} + Ce^{-|x_1|/2}$$

and[4]

$$0 \le G''_\alpha(x_1) \le C\sin\alpha\frac{|\log|x_1||}{1 + x_1^4|\log|x_1||} + Ce^{-|x_1|/2}.$$

Proof. By definition, the Fourier transform of G_α is given by

$$\mathcal{F}G_\alpha(\xi) = \frac{1}{\sqrt{2\pi}}\frac{1}{\xi^2 + 1 + |\xi|\sin\alpha}, \quad \xi \in \mathbb{R},$$

which immediately implies that $G_\alpha \in H^1(\mathbb{R})$ with

$$\|G_\alpha\|_{H^1(\mathbb{R})} \le C_1$$

for a constant $C_1 > 0$ independent of α. As $LG_\alpha = \delta_0$, we deduce that $G''_\alpha \in \delta_0 + L^2(\mathbb{R})$ (as a distribution). As a function, however, G_α is smooth at every $x_1 \neq 0$ with

$$G'_\alpha(x_1) = -\frac{x_1}{|x_1|}\frac{\sin\alpha}{\pi}\int_0^\infty \frac{t^2e^{-t|x_1|}}{t^2\sin^2\alpha + (t^2-1)^2}\,dt$$

and

$$G''_\alpha(x_1) = \frac{\sin\alpha}{\pi}\int_0^\infty \frac{t^3e^{-t|x_1|}}{t^2\sin^2\alpha + (t^2-1)^2}\,dt.$$

[3]The linearisation of (14) about $\phi_0 = \alpha$ is then given by $L\left(x_1 \mapsto \sin^2\alpha\psi\left(\frac{x_1}{\sin\alpha}\right)\right)$.

[4]In the sense of distributions, we have $G''_\alpha \in \delta_0 + L^2(\mathbb{R})$, so we estimate the diffuse part of G''_α here (still denoted G''_α).

Step 1: estimates for $|x_1| \geq 1$. We have

$$\int_0^{1/2} \frac{te^{-t|x_1|}}{t^2\sin^2\alpha + (t^2-1)^2}\,dt \leq 4\int_0^\infty te^{-t|x_1|}\,dt = \frac{4}{x_1^2}\int_0^\infty se^{-s}\,ds.$$

If $\alpha \leq \frac{\pi}{6}$, then

$$\int_{1/2}^{1-\sin\alpha} \frac{te^{-t|x_1|}}{t^2\sin^2\alpha + (t^2-1)^2}\,dt \leq e^{-|x_1|/2}\int_{-\infty}^{1-\sin\alpha} \frac{dt}{(t-1)^2} = \frac{e^{-|x_1|/2}}{\sin\alpha}$$

and

$$\int_{1+\sin\alpha}^\infty \frac{te^{-t|x_1|}}{t^2\sin^2\alpha + (t^2-1)^2}\,dt \leq e^{-|x_1|}\int_{1+\sin\alpha}^\infty \frac{dt}{(t-1)^2} = \frac{e^{-|x_1|}}{\sin\alpha}$$

as well. Moreover,

$$\int_{1-\sin\alpha}^{1+\sin\alpha} \frac{te^{-t|x_1|}}{t^2\sin^2\alpha + (t^2-1)^2}\,dt \leq \frac{e^{-|x_1|/2}}{\sin^2\alpha}\int_{1-\sin\alpha}^{1+\sin\alpha} \frac{dt}{t} \leq \frac{4e^{-|x_1|/2}}{\sin\alpha}.$$

If $\alpha > \frac{\pi}{6}$, then we observe instead that

$$\int_{1/2}^\infty \frac{te^{-t|x_1|}}{t^2\sin^2\alpha + (t^2-1)^2}\,dt \leq e^{-|x_1|/2}\int_{1/2}^\infty \frac{t}{t^2/4 + (t^2-1)^2}\,dt.$$

The integral on the right-hand side converges, and the inequalities for G_α follow immediately. For G'_α and G''_α, we can use the same arguments when $|x_1| \geq 1$.

Step 2: estimates for $|x_1| < 1$. For G_α, we know that $\|G_\alpha\|_{H^1(\mathbb{R})}$ is bounded uniformly in α. We conclude that $|G_\alpha(x_1)|$ is bounded uniformly in $\alpha \in (0, \frac{\pi}{2}]$ and $x_1 \in [-1, 1]$.

For G'_α and G''_α, we first observe that

$$|G'_\alpha(x_1)| \leq 2G_\alpha(x_1) + \frac{4\sin\alpha}{\pi}\int_2^\infty \frac{dt}{t^2} = 2G_\alpha(x_1) + \frac{2\sin\alpha}{\pi}$$

and

$$|G''_\alpha(x_1)| \leq 4G_\alpha(x_1) + \frac{4\sin\alpha}{\pi}\int_2^\infty t^{-1}e^{-t|x_1|}\,dt.$$

Since

$$\int_2^\infty t^{-1}e^{-t|x_1|}\,dt \leq \int_2^{2/|x_1|} \frac{dt}{t} + \int_2^\infty e^{-s}\frac{ds}{s} = \log\frac{1}{|x_1|} + \int_2^\infty e^{-s}\frac{ds}{s},$$

the desired inequalities follow for $|x_1| < 1$ as well. □

A considerable part of the subsequent analysis is based on the decay behaviour of G_α and its derivatives, together with the following principle: if $G, \psi \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, then

$$(G * \psi)(x_1) = \int_{x_1/2}^\infty (G(t)\psi(x_1 - t) + G(x_1 - t)\psi(t))\,dt;$$

therefore,

$$|(G * \psi)(x_1)| \leq \|G\|_{L^\infty(x_1/2,\infty)}\|\psi\|_{L^1(\mathbb{R})} + \|G\|_{L^1(\mathbb{R})}\|\psi\|_{L^\infty(x_1/2,\infty)}. \tag{24}$$

5.3 Polynomial decay for $h < 1$

For $h < 1$, we will prove polynomial decay for minimisers of E_h in $\mathcal{A}_h(d)$ for $d \in \{\frac{\alpha}{\pi}, 1 - \frac{\alpha}{\pi}\}$. The following decay estimates improve the results of Chermisi-Muratov [2, Lemma 5]. In particular, we prove cubic and quartic decay of f' and f'', respectively, as well as a new L^1-estimate for f, which is fundamental for the proofs of our main results stated in Section 1.3.

Theorem 5.2. *There exist universal constants $c, C > 0$ with the following property. For every $h = \cos\alpha$ with $\alpha \in (0, \frac{\pi}{2}]$, there exists a unique increasing, odd function $\phi\colon \mathbb{R} \to \mathbb{R}$ such that $m = (\cos\phi, \sin\phi)$ is a minimiser of E_h in $\mathcal{A}_h(\alpha/\pi)$. Furthermore, the function $f = \cos\phi - \cos\alpha$ satisfies*

$$0 < f(x_1) \leq \frac{C}{x_1^2}, \quad |f'(x_1)| \leq \frac{C}{x_1^3}, \quad |f''(x_1)| \leq \frac{C}{x_1^4}, \quad \text{and} \quad |\Lambda f(x_1)| \leq \frac{C\alpha}{x_1^2} \quad \text{for all } x_1 \geq \frac{c}{\alpha}$$

and also

$$\|f\|_{L^1(\mathbb{R})} \leq C\alpha.$$

Proof. We use various universal constants in this proof, and we will abuse notation and indiscriminately use the symbol C for most of them. The existence of a unique symmetric minimiser follows by symmetrization via rearrangement as proved in the works of Melcher [15] and Chermisi-Muratov [2] (see also Lemma 3.2 above). Moreover, ϕ is increasing with $\phi(\mathbb{R}) = (-\alpha, \alpha)$. By the symmetry, the function ϕ is odd. Thus it suffices to prove the inequalities. To this end, we first rescale the solutions.

Step 1: rescaling. Set $f = \cos\phi - \cos\alpha$ and

$$g(x_1) = \frac{1}{\sin^2\alpha} f\left(\frac{x_1}{\sin\alpha}\right).$$

As $0 < f \leq 1 - \cos\alpha$ in $\mathbb{R}$, we deduce that $0 < g \leq 1$. Moreover, as f satisfies (16) away from $x_1 = 0$, we know that g is a solution of the equation

$$g'' = -\frac{(g')^2(g\sin^2\alpha + \cos\alpha)}{1 - 2g\cos\alpha - g^2\sin^2\alpha} + (g - \sin\alpha\Lambda g)(1 - 2g\cos\alpha - g^2\sin^2\alpha), \quad x_1 \neq 0.$$

Define the operator L as in (22). Then we can write the equation in the form

$$Lg = A(g')^2 + gB(g - \sin\alpha\Lambda g) \quad \text{in } \mathbb{R}\setminus\{0\}, \tag{25}$$

where

$$B = 2\cos\alpha + g\sin^2\alpha = \cos\alpha + \cos\phi(\frac{\cdot}{\sin\alpha})$$

and

$$A = \frac{g\sin^2\alpha + \cos\alpha}{1 - gB} = \sin^2\alpha \frac{\cos\phi(\frac{\cdot}{\sin\alpha})}{\sin^2\phi(\frac{\cdot}{\sin\alpha})} \quad \text{in } \mathbb{R}\setminus\{0\}.$$

The function B is bounded (with $|B| \leq 2$ in $\mathbb{R}$), whereas A is unbounded for every $\alpha \in (0, \frac{\pi}{2}]$ (since $A(x_1) \to \infty$ as $x_1 \to 0$) and $A > 0$ for $x_1 \neq 0$. However, for any x_1 such that $|\phi(\frac{x_1}{\sin\alpha})| \geq \frac{\alpha}{2}$, we have $A(x_1) \leq C$.

Step 2: prove L^2-estimates. We want to show that

$$\left(\int_{-\infty}^{\infty} A(g')^2\, dx_1\right)^{1/2} + \|g\|_{L^2(\mathbb{R})} + \alpha\|g'\|_{L^2(\mathbb{R})} \leq C. \tag{26}$$

To this end, we first compute

$$A(x_1)(g'(x_1))^2 = \frac{\cos\phi(\frac{x_1}{\sin\alpha})}{\sin^4\alpha}\left(\phi'\left(\frac{x_1}{\sin\alpha}\right)\right)^2, \quad x_1 \neq 0.$$

Therefore,

$$\int_{-\infty}^{\infty} A(g')^2\, dx_1 \leq \frac{1}{\sin^3\alpha}\int_{-\infty}^{\infty} (\phi')^2\, dx_1 \leq \frac{2E_h(m)}{\sin^3\alpha},$$

where $m = (\cos\phi, \sin\phi)$. Furthermore, we compute

$$\|g\|_{L^2(\mathbb{R})}^2 = \frac{\|f\|_{L^2(\mathbb{R})}^2}{\sin^3\alpha} \leq \frac{2E_h(m)}{\sin^3\alpha}$$

and similarly

$$\|g'\|_{L^2(\mathbb{R})}^2 = \frac{\|f'\|_{L^2(\mathbb{R})}^2}{\sin^5\alpha} \leq \frac{2E_h(m)}{\sin^5\alpha}.$$

Using Lemma 2.1, we obtain (26).

Step 3: prove preliminary pointwise estimates. Next we want to establish the following inequalities:

$$0 < g(x_1) \leq \frac{C}{\sqrt{|x_1|}} \qquad \text{for any } x_1 \neq 0, \tag{27}$$

$$|g'(x_1)| \leq \frac{C}{\sqrt{\alpha}|x_1|} \qquad \text{for any } x_1 \neq 0, \tag{28}$$

$$|(\Lambda g)(x_1)| \leq \frac{C}{\sqrt{\alpha}|x_1|} \qquad \text{for } |x_1| > \sin\alpha. \tag{29}$$

For the proof of (27), we will in fact show that $g(x_1) \le x_1^{-1/2}\|g\|_{L^2(\mathbb{R})}$ for $x_1 > 0$. The inequality then follows by the symmetry and (26). Assume, for contradiction, that there exists $x_1 > 0$ with $g(x_1) > x_1^{-1/2}\|g\|_{L^2(\mathbb{R})}$. Then for every $t \in (0, x_1)$, we have $g(t) \ge g(x_1) > x_1^{-1/2}\|g\|_{L^2(\mathbb{R})}$, because g is non-increasing. Therefore,

$$\int_0^{x_1} g^2\,dt > \|g\|_{L^2(\mathbb{R})}^2 \int_0^{x_1} \frac{1}{x_1}\,dt = \|g\|_{L^2(\mathbb{R})}^2,$$

which is a contradiction.

As $\phi(0) = 0$ and ϕ is increasing, we have $0 < \phi < \alpha$ for $x_1 > 0$. Thus we may use Lemma 3.3 (for $a = 0$ and with $R/\sin\alpha$ instead of R) and Lemma 2.1 to conclude that

$$\int_{\frac{R}{\sin\alpha}}^\infty \left((\phi'')^2 + (\phi')^2\sin^2\phi + (\phi')^4\right)\,dx_1 \le \frac{C\sin^2\alpha\,E_h(m)}{R^2} \le \frac{C\sin^5\alpha}{R^2} \quad \text{for any } R > 0.$$

Hence

$$\int_R^\infty \left(\sin^2\alpha(g'')^2 + (g')^2\right)\,dx_1 \le \frac{C}{R^2} \quad \text{for any } R > 0. \tag{30}$$

In particular, the Cauchy-Schwartz inequality implies, for every $t > R$, that

$$\sin\alpha\left|(g'(R))^2 - (g'(t))^2\right| \le 2\sin\alpha\int_R^t |g'g''|\,dx_1 \le \frac{C}{R^2}.$$

As $g' \in L^2(\mathbb{R})$, we know that $\liminf_{t\to\infty}|g'(t)| = 0$; so (28) follows. We finally apply Lemmas 4.5 and 2.1 to obtain

$$|(\Lambda g)(x_1)| = \frac{1}{\sin^3\alpha}|(\Lambda f)(\frac{x_1}{\sin\alpha})| \le \frac{C}{\sqrt{\alpha}|x_1|} \quad \text{for } |x_1| > \sin\alpha,$$

which is (29).

We also note that as a consequence of (27), there exists a constant $a \ge 1$ (independent of α) such that

$$\phi\left(\frac{x_1}{\sin\alpha}\right) \ge \frac{\alpha}{2} \text{ (and hence } A(x_1) \le C\text{) whenever } x_1 \ge a. \tag{31}$$

Step 4: improve the decay. We now show that (27) can be improved as follows:

$$g(x_1) \le \frac{C}{|x_1|} \quad \text{for } |x_1| \ge 2a.$$

For this purpose, we use the fact that $g = G_\alpha * Lg$. As $|B| \le 2$ and $g > 0$ in $\mathbb{R}$, we have

$$g \le G_\alpha * A(g')^2 + 2G_\alpha * g^2 + 2\alpha G_\alpha * g|\Lambda g| \quad \text{for } x_1 \ne 0. \tag{32}$$

Applying an inequality of the type of (24), we find

$$\begin{aligned}(G_\alpha * A(g')^2)(x_1) &= \int_{x_1/2}^\infty \Big(G_\alpha(t)[A(g')^2](x_1 - t) + G_\alpha(x_1 - t)[A(g')^2](t)\Big)\,dt \\ &\le \|G_\alpha\|_{L^\infty(x_1/2,\infty)}\|A(g')^2\|_{L^1(\mathbb{R})} + \|G_\alpha\|_{L^\infty(\mathbb{R})}\|A(g')^2\|_{L^1(x_1/2,\infty)}.\end{aligned}$$

By (31), we have $|A(x_1)| \le C$ for $|x_1| \ge a$. Hence when $|x_1| \ge 2a$, Lemma 5.2, together with (26) and (30), implies that

$$(G_\alpha * A(g')^2)(x_1) \le \frac{C}{x_1^2}.$$

Similarly, we use (24) to estimate the other two terms in (32). Owing to (26) and (27), we obtain

$$|(G_\alpha * g^2)(x_1)| \le \frac{C}{|x_1|}, \quad x_1 \ne 0.$$

Because

$$\|g\Lambda g\|_{L^1(\mathbb{R})} \le \|g\|_{L^2(\mathbb{R})}\|g'\|_{L^2(\mathbb{R})} \le \frac{C}{\alpha}$$

by (26) and

$$g(x_1)|(\Lambda g)(x_1)| \le \frac{C}{\sqrt{\alpha}|x_1|^{3/2}}, \quad x_1 \ne 0,$$

by (27) and (29), we also have

$$|\alpha(G_\alpha * g|\Lambda g|)(x_1)| \le \frac{C}{|x_1|^{3/2}}, \quad x_1 \ne 0.$$

Therefore, the desired decay for g follows when $|x_1| \ge 2a$.

Step 5: conclusion. We can use the conclusion of Step 4 to improve the above estimates again. Namely, we find that

$$|(G_\alpha * g^2)(x_1)| \leq \frac{C}{x_1^2}$$

and

$$|\alpha(G_\alpha * g\Lambda g)(x_1)| \leq \frac{C}{x_1^2}$$

for $|x_1| \geq 4a$. Hence

$$0 < g(x_1) \leq \frac{C}{x_1^2} \quad \text{for } |x_1| \geq 4a. \tag{33}$$

Using the formula

$$g' = G'_\alpha * Lg \tag{34}$$

and taking advantage of (33), we repeat the arguments from Step 4 to obtain, for $|x_1| \geq 8a$,

$$|(G'_\alpha * A(g')^2)(x_1)| \leq \frac{C}{x_1^2}, \quad |(G_\alpha * g^2)(x_1)| \leq \frac{C}{|x_1|^3}, \quad |\alpha(G_\alpha * g|\Lambda g|)(x_1)| \leq \frac{C}{|x_1|^3}.$$

Therefore,

$$|g'(x_1)| \leq \frac{C}{|x_1|^2} \quad \text{for } |x_1| \geq 8a.$$

Using this estimate, we obtain

$$\int_{|x_1|/2}^\infty A(g')^2\, dt \leq \frac{C}{|x_1|^3}, \quad |x_1| \geq 16a,$$

so that $|(G'_\alpha * A(g')^2)(x_1)| \leq \frac{C}{x_1^3}$, and finally,

$$|g'(x_1)| \leq \frac{C}{|x_1|^3} \quad \text{for } |x_1| \geq 16a.$$

As $g'' = G''_\alpha * Lg$, the same method[5] implies, for $|x_1| \geq 32a$, that

$$|(G''_\alpha * A(g')^2)(x_1)| \leq \|G''_\alpha\|_{L^\infty(x_1/2,\infty)} \|A(g')^2\|_{L^1(\mathbb{R})} + \|G''_\alpha\|_{L^1(\mathbb{R})} \|A(g')^2\|_{L^\infty(x_1/2,\infty)} \leq \frac{C}{x_1^4},$$

$$|(G''_\alpha * g^2)(x_1)| \leq \frac{C}{|x_1|^4}, \quad |\alpha(G''_\alpha * g|\Lambda g|)(x_1)| \leq \frac{C}{|x_1|^3}.$$

This in turn yields

$$|g''(x_1)| \leq \frac{C}{|x_1|^3}, \quad |x_1| \geq 32a.$$

In order to obtain the desired quartic power decay of g'', we need to improve the estimate of $g|\Lambda g|$. To this end, we use Proposition 4.1 (applied with p sufficiently close to 1, $q = 1$ and $R = x_1/2$). We find that

$$|\Lambda g(x_1)| \leq \frac{C\big|\log|x_1|\big|}{|x_1|^2}$$

for $|x_1| \geq 64a$, using the fact that $\|g\|_{L^1(\mathbb{R})} \leq C$ (because g is bounded and satisfies (33)). Hence

$$g(x_1)|\Lambda g(x_1)| \leq \frac{C\big|\log|x_1|\big|}{|x_1|^4}, \quad \text{as well as} \quad |\alpha(G''_\alpha * g|\Lambda g|)(x_1)| \leq \frac{C\big|\log|x_1|\big|}{|x_1|^4},$$

which yields

$$|g''(x_1)| \leq \frac{C\big|\log|x_1|\big|}{|x_1|^4}$$

for $|x_1| \geq 128a$. Applying Proposition 4.1 again, we obtain

$$|\Lambda g(x_1)| \leq \frac{C}{x_1^2},$$

leading to

$$|g''(x_1)| \leq \frac{C}{x_1^4} \quad \text{for } |x_1| \geq 256a.$$

Now the inequalities for f follow by rescaling. □

[5]Note that G''_α does not belong to L^∞ (by Lemma 5.2) so that we can only use L^1 estimates near $x_1 = 0$.

We also state a similar statement for minimisers in the set $\mathcal{A}_h(1-\alpha/\pi)$, but we are not concerned about the dependence of the constants on α here.

Theorem 5.3. *Suppose that $h = \cos\alpha$ for $\alpha \in (0, \frac{\pi}{2}]$. Then there exists an increasing function $\phi\colon \mathbb{R} \to \mathbb{R}$ such that $\phi - \pi$ is odd and $m = (\cos\phi, \sin\phi)$ is a minimiser of E_h in $\mathcal{A}_h(1-\alpha/\pi)$. Furthermore, the function $f = \cos\alpha - \cos\phi$ satisfies*

$$\limsup_{x_1\to\pm\infty} \left(x_1^2|f(x_1)| + |x_1|^3|f'(x_1)| + x_1^4|f''(x_1)| + x_1^2|\Lambda f(x_1)|\right) < \infty.$$

Proof. This can be proved with the same arguments.

□

6 Concentration compactness

6.1 Strategy

We want to prove Theorem 1.1 and Theorem 1.3 through the analysis of minimising sequences for E_h in the sets $\mathcal{A}_h(d)$. Similarly to many other variational problems involving topological information, the main difficulty in proving existence of minimisers is a possible 'escape to infinity' of a topologically non-trivial part of the members of a minimising sequence. (This corresponds to the 'dichotomy' case in the concentration-compactness framework of Lions [14].) In order to prevent this, we want to improve Proposition 2.3 by showing that

$$\mathcal{E}_h(d) < \mathcal{E}_h(d_1) + \mathcal{E}_h(d_2) \tag{35}$$

for all appropriate decompositions $d = d_1 + d_2$ into smaller winding numbers. We will achieve this by constructing a magnetisation profile of winding number d from two energy minimisers in $\mathcal{A}_h(d_1)$ and $\mathcal{A}_h(d_2)$ and estimating the energy (see Theorems 7.1 and 7.2 below). This is where the analysis of the Euler-Lagrange equation from the previous sections, and in particular the decay at $\pm\infty$, will be crucial.

In this chapter, we show how inequalities of the type (35) give rise to minimisers in $\mathcal{A}_h(d)$. Due to the symmetry proved in Lemma 3.2, we may in fact work with a somewhat weaker hypothesis than expected.

6.2 Statement

We formulate the following result for $d \in \mathbb{N} - \frac{\alpha}{\pi}$ only (which in the case $h > 1$ means $d \in \mathbb{N}$). Although a similar statement would always be true, we do not expect that the hypothesis of Theorem 6.1 will be satisfied if $h < 1$ and $d \in \mathbb{N}$ or $d \in \mathbb{N} + \frac{\alpha}{\pi}$. Of course we automatically obtain statements for $d \in \frac{\alpha}{\pi} - \mathbb{N}$ as well.

Theorem 6.1 (Concentration compactness)**.** *Suppose that $d = \ell - \alpha/\pi$ for some $\ell \in \mathbb{N}$ such that*

$$\mathcal{E}_h(d) < 2\mathcal{E}_h(d') + \mathcal{E}_h(d - 2d')$$

for $d' = 1-\alpha/\pi, 1, 2-\alpha/\pi, 2, \ldots, \ell/2 - 1, \ell/2 - \alpha/\pi$ if ℓ is even and for $d' = 1-\alpha/\pi, 1, 2-\alpha/\pi, 2, \ldots, (\ell-1)/2 - \alpha/\pi, (\ell-1)/2$ if ℓ is odd. Then E_h attains its infimum in $\mathcal{A}_h(d)$.

Proof. We divide the proof in several steps.

Step 1: pick a minimising sequence. Consider a minimising sequence $(m^j)_{j\in\mathbb{N}}$ of E_h in $\mathcal{A}_h(d)$. By Lemma 3.2, we may assume that each m^j is symmetric. In particular, we have $m^j(0) = ((-1)^\ell, 0)$ for every $j \in \mathbb{N}$. It is clear that a subsequence converges weakly in $H^1_{\mathrm{loc}}(\mathbb{R}; \mathbb{S}^1)$. We may assume without loss of generality that this applies to the whole sequence, i.e., that $m^j \rightharpoonup m$ weakly in $H^1_{\mathrm{loc}}(\mathbb{R}; \mathbb{S}^1)$ for some $m \in H^1_{\mathrm{loc}}(\mathbb{R}; \mathbb{S}^1)$. Then m is symmetric as well with $m(0) = ((-1)^\ell, 0)$. It is also clear that the energy is lower semicontinuous with respect to such convergence. Thus

$$E_h(m) \le \liminf_{j\to\infty} E_h(m^j) = \mathcal{E}_h(d).$$

In particular, we have $\lim_{x_1\to\pm\infty} m_1(x_1) = k$, and the winding number $\tilde{d} = \deg(m)$ is well-defined and belongs to $\mathbb{Z} + \{0, \pm\alpha/\pi\}$. Because of the symmetry and because $m(0) = (\pm 1, 0)$, we have $\tilde{d} \neq 0$. If we can show that $\tilde{d} = d$, then it follows that $m \in \mathcal{A}_h(d)$ and that m is a minimiser of E_h in this set, which then concludes the proof. The aim of the next steps is to show that $\tilde{d} = d$.

Step 2: some properties of the minimising sequence. First note that in the case $h < 1$, we obviously have $(m_1 - h)^2 \le 2W(m)$, whereas in the case $h > 1$, we have

$$(m_1 - 1)^2 \le 2(1 - m_1) \le 2(1 - m_1) + \frac{1}{h-1}(1 - m_1)^2 = \frac{2}{h-1}W(m).$$

Hence $m_1 - k \in L^2(\mathbb{R})$, which implies that

$$\lim_{j\to\infty} \int_{[-2j,-j]\cup[j,2j]} (m_1 - k)^2 \, dx_1 = 0.$$

Without loss of generality, we may assume that

$$\int_{-2j}^{2j} |m^j - m|^2 \, dx_1 \le \frac{1}{j^5} \tag{36}$$

for every $j \in \mathbb{N}$ (as we can always select a subsequence with this property and then relabel the indices). Then

$$\lim_{j\to\infty} \int_{[-2j,-j]\cup[j,2j]} (m_1^j - k)^2 \, dx_1 = 0 \tag{37}$$

as well. Since

$$\limsup_{j\to\infty} \int_{-\infty}^{\infty} |(m^j)'|^2 \, dx_1 < \infty \tag{38}$$

and

$$\left\| \frac{d}{dx_1} |m - m^j|^2 \right\|_{L^1(-2j,2j)} \le 2\|m' - (m^j)'\|_{L^2(\mathbb{R})} \|m - m^j\|_{L^2(-2j,2j)},$$

then (36) and (38), together with the fact that $m(0) = m^j(0)$, yield:

$$\limsup_{j\to\infty} j\|m^j - m\|_{L^\infty(-2j,2j)} = 0. \tag{39}$$

Similarly, as there exist $t_j \in (j, 2j)$ and $s_j \in (-2j, -j)$ such that $m_1^j(t_j), m_1^j(s_j) \to k$ as $j \to \infty$, we deduce

$$\lim_{j\to\infty} \|m_1^j - k\|_{L^\infty([-2j,-j]\cup[j,2j])} = 0. \tag{40}$$

Moreover, it follows from (39) that

$$\lim_{j\to\infty} \int_{-2j}^{2j} (m^j)^\perp \cdot (m^j)' \, dx_1 = 2\pi\tilde{d}. \tag{41}$$

Indeed, let ϕ and ϕ^j be continuous liftings of m and m^j, respectively. Due to (39), we may assume that $\|\phi^j - \phi\|_{L^\infty(-2j,2j)} \to 0$, too. As

$$\int_{-2j}^{2j} (m^j)^\perp \cdot (m^j)' \, dx_1 = \phi^j(2j) - \phi^j(-2j),$$

we conclude that (41) holds true.

Step 3: cut-off. Choose $\eta \in C^\infty(\mathbb{R})$ with $\eta \equiv 0$ in $(-\infty, 0]$, $\eta \equiv 1$ in $[1, \infty)$, and $0 < \eta < 1$ in $(0, 1)$. For $j \in \mathbb{Z} \setminus \{0\}$, let $\hat{\eta}_j(x_1) = \eta(4x_1/j - 7)$ and $\tilde{\eta}_{|j|}(x_1) = \eta(4x_1/j - 4) + \eta(-4x_1/j - 4)$. Now define, for $j \in \mathbb{Z} \setminus \{0\}$, the functions

$$\hat{m}_1^j = \hat{\eta}_j m_1^{|j|} + (1 - \hat{\eta}_j)k$$

(cut off to the left of $2j$ if $j > 0$ and to the right of $-2j$ if $j < 0$) and

$$\tilde{m}_1^j = (1 - \tilde{\eta}_j)m_1^j + \tilde{\eta}_j k, \quad j > 0$$

(cut off outside of $(-j, j)$). Note that for $j \in \mathbb{N}$, the functions $\tilde{m}_1^j - k$, $\hat{m}_1^{-j} - k$ and $\hat{m}_1^j - k$ have disjoint support. For $j \in \mathbb{Z}$ with $|j|$ sufficiently large, owing to (40), there exist functions $\hat{m}_2^j \colon \mathbb{R} \to [-1, 1]$ such that $\hat{m}_2^j(x_1) = m_2^{|j|}(x_1)$ if $j > 0$ and $x_1 \ge 2j$ or $j < 0$ and $x_1 \le -2j$ and such that $\hat{m}^j = (\hat{m}_1^j, \hat{m}_2^j)$ takes values in $\mathbb{S}^1$. Similarly, for $j \in \mathbb{N}$ sufficiently large, there exists a function $\tilde{m}_2^j \colon \mathbb{R} \to [-1, 1]$ such that $\tilde{m}_2^j(x_1) = m_2^j(x_1)$ for $|x_1| \le j$ and such that $\tilde{m}^j = (\tilde{m}_1^j, \tilde{m}_2^j)$ takes values in $\mathbb{S}^1$. The aim of the next steps is to prove that

$$\limsup_{j\to\infty} \left(E_h(\tilde{m}^j) + E_h(\hat{m}^j) + E_h(\hat{m}^{-j}) \right) \le \mathcal{E}_h(d). \tag{42}$$

Step 4: estimate the anisotropy and exchange energy. Because we have the pointwise inequalities $W(m^j) \geq W(\hat{m}^j)$ and $W(m^j) \geq W(\tilde{m}^j)$, it is clear that

$$\limsup_{j\to\infty} \int_{-\infty}^{\infty} \left(W(\tilde{m}^j) + W(\hat{m}^j) + W(\hat{m}^{-j}) - W(m^j) \right) dx_1 \leq 0.$$

In order to estimate the exchange energy, note first that in $[-2j,-j] \cup [j,2j]$, we have

$$\left((\tilde{m}_1^j)' \right)^2 = (1-\tilde{\eta}_j)^2 \left((m_1^j)' \right)^2 - 2(1-\tilde{\eta}_j)\tilde{\eta}_j'(m_1^j - k)(m_1^j)' + (\tilde{\eta}_j')^2 (m_1^j - k)^2$$

and

$$\left((\hat{m}_1^{\pm j})' \right)^2 = \hat{\eta}_{\pm j}^2 \left((m_1^j)' \right)^2 + 2\hat{\eta}_{\pm j}\hat{\eta}_{\pm j}'(m_1^j - k)(m_1^j)' + (\hat{\eta}_{\pm j}')^2 (m_1^j - k)^2.$$

In the case $h < 1$, the integrals of the last two terms in each identity over $(-2j,-j) \cup (j,2j)$ will tend to 0 as $j \to \infty$ due to (37), (38), and the inequalities $\|(\hat{\eta}^{\pm j})'\|_{L^\infty(\mathbb{R})} + \|(\tilde{\eta}^j)'\|_{L^\infty(\mathbb{R})} \leq \|\eta'\|_{L^\infty(\mathbb{R})}/j$. Because of (40), we have

$$1 - (\tilde{m}_1^j)^2 \to \sin^2\alpha \quad \text{and} \quad 1 - (\hat{m}_1^{\pm j})^2 \to \sin^2\alpha$$

uniformly in $[-2j,-j] \cup [j,2j]$, as well as $1 - (m_1^j)^2 \to \sin^2\alpha$. It follows that

$$\limsup_{j\to\infty} \int_{-\infty}^{\infty} \left(\frac{\left((\tilde{m}_1^j)'\right)^2}{1-(\tilde{m}_1^j)^2} + \frac{\left((\hat{m}_1^j)'\right)^2}{1-(\hat{m}_1^j)^2} + \frac{\left((\hat{m}_1^{-j})'\right)^2}{1-(\hat{m}_1^{-j})^2} - \frac{\left((m_1^j)'\right)^2}{1-(m_1^j)^2} \right) dx_1 \leq 0. \tag{43}$$

In the case $h > 1$, we note that

$$1 - \tilde{m}_1^j = (1-\tilde{\eta}_j)(1-m_1^j) \quad \text{and} \quad 1 - \hat{m}_1^j = \hat{\eta}_j(1-m_1^j)$$

for $j > 0$. Due to the uniform convergence of $1 + \tilde{m}_1^j \to 2$, $1 + \hat{m}_1^j \to 2$, as well as $1 + m_1^j \to 2$ in $[j,2j]$ as $j \to \infty$, estimating the exchange energy reduces to analysing the following terms:

$$\frac{\left((\tilde{m}_1^j)'\right)^2}{1-\tilde{m}_1^j} = (1-\tilde{\eta}_j)\frac{\left((m_1^j)'\right)^2}{1-m_1^j} + 2\tilde{\eta}_j'(m_1^j)' - \frac{(\tilde{\eta}_j')^2}{1-\tilde{\eta}_j}(m_1^j - 1)$$

and

$$\frac{\left((\hat{m}_1^j)'\right)^2}{1-\hat{m}_1^j} = \hat{\eta}_j\frac{\left((m_1^j)'\right)^2}{1-m_1^j} - 2\hat{\eta}_j'(m_1^j)' - \frac{(\hat{\eta}_j')^2}{\hat{\eta}_j}(m_1^j - 1).$$

By l'Hôpital's rule,

$$\lim_{x_1 \nearrow 1} \frac{(\eta'(x_1))^2}{1-\eta(x_1)} = -2\eta''(1) = 0,$$

and thus the function $\frac{(\tilde{\eta}_j')^2}{1-\tilde{\eta}_j}$ is bounded and supported on $[-2j,-j] \cup [j,2j]$. Similar arguments apply to $\hat{\eta}_j$. Moreover, we obviously have

$$\|\tilde{\eta}_j'(m_1^j)'\|_{L^1(\mathbb{R})} \leq \frac{4}{\sqrt{j}} \|\eta'\|_{L^2(\mathbb{R})} \|(m^j)'\|_{L^2(\mathbb{R})} \to 0$$

as $j \to \infty$. Since the corresponding estimates hold in $[-2j,-j]$ for $\hat{m}_1^{-j}$ instead of $\hat{m}_1^j$, we conclude that (43) holds true in the case $h > 1$, too.

Step 5: estimate the stray field energy. Next we want to estimate $\|\tilde{m}_1^j - k\|_{\dot{H}^{1/2}(\mathbb{R})}$ and $\|\hat{m}_1^j - k\|_{\dot{H}^{1/2}(\mathbb{R})}$. Let $\tilde{v}_j = V(\tilde{m}^j)$ and $\hat{v}_{\pm j} = V(\hat{m}^{\pm j})$ as defined in (10). Furthermore, let $v_j = V(m^j)$ and $w_j = v_j - \tilde{v}_j - \hat{v}_j - \hat{v}_{-j}$ for $j \in \mathbb{N}$. Then $w_j(\,\cdot\,, 0) \to 0$ in $L^2(\mathbb{R})$ by (37), while $w_j'(\,\cdot\,, 0)$ remains bounded in $L^2(\mathbb{R})$ by (38). Thus standard interpolation between $\dot{H}^1(\mathbb{R})$ and $L^2(\mathbb{R})$ implies that

$$\lim_{j\to\infty} \int_{\mathbb{R}^2_+} |\nabla w_j|^2 \, dx = \lim_{j\to\infty} \|w_j(\cdot,0)\|^2_{\dot{H}^{1/2}(\mathbb{R})} = 0.$$

Hence

$$\lim_{j\to\infty} \int_{\mathbb{R}^2_+} \left(|\nabla \tilde{v}_j + \nabla \hat{v}_j + \nabla \hat{v}_{-j}|^2 - |\nabla v_j|^2 \right) dx = 0 \tag{44}$$

by the triangle inequality. Moreover, as the sequences $(\tilde{m}_1^j - k)_{j\in\mathbb{N}}$ and $(\hat{m}_1^j - k)_{j\in\mathbb{Z}\setminus\{0\}}$ are bounded in $H^1(\mathbb{R})$, it follows that

$$\limsup_{j\to\infty} \int_{\mathbb{R}^2_+} \left(|\nabla \tilde{v}_j|^2 + |\nabla \hat{v}_j|^2 + |\nabla \hat{v}_{-j}|^2 \right) dx < \infty.$$

By Lemma 4.2, integration by parts yields

$$\lim_{j\to\infty} \int_{\mathbb{R}^2_+} \nabla \tilde{v}_j \cdot \nabla \hat{v}_{\pm j}\, dx \overset{(12)}{=} - \lim_{j\to\infty} \int_{\mathbb{R}} \Lambda(\tilde{m}_1^j - k)(\hat{m}_1^{\pm j} - k)\, dx_1 \overset{(18)}{=} 0$$

and

$$\lim_{j\to\infty} \int_{\mathbb{R}^2_+} \nabla \hat{v}_j \cdot \nabla \hat{v}_{-j}\, dx = 0.$$

Therefore, in view of (44), we obtain

$$\lim_{j\to\infty} \int_{\mathbb{R}^2_+} \left(|\nabla \tilde{v}_j|^2 + |\nabla \hat{v}_j|^2 + |\nabla \hat{v}_{-j}|^2 - |\nabla v_j|^2 \right) dx = 0.$$

Now (42) is proved.

Step 6: conclusion. We conclude from (40) and (41) that $\deg(\tilde{m}^j) = \tilde{d}$ whenever j is sufficiently large. Then by the symmetry, we have $\deg(\hat{m}^{\pm j}) = \frac{1}{2}(d - \tilde{d})$. Because of (42), we have

$$\mathcal{E}_h(|\tilde{d}|) + 2\mathcal{E}_h\left(\frac{1}{2}|d - \tilde{d}|\right) \le \mathcal{E}_h(d).$$

It is clear that $\mathcal{E}_h(\delta) > 0$ whenever $\delta \neq 0$. Therefore, we can draw the following conclusions from the above inequality. First, we have already seen that $\tilde{d} \neq 0$. Second, we conclude that $\tilde{d} \le d$. (Otherwise, Proposition 2.2 would imply that $\mathcal{E}_h(|\tilde{d}|) \ge \mathcal{E}_h(d)$, which is inconsistent with the inequality.) Third, we conclude that $\tilde{d} \ge 0$. (Otherwise, set $d_1 = \frac{1}{2}(d - \tilde{d})$ and choose the largest number $d_2 \le \frac{1}{2}(d - \tilde{d})$ such that Proposition 2.3 applies to d_1 and d_2. Then $d_1 + d_2 \ge d$ and hence $\mathcal{E}_h(d) \le \mathcal{E}_h(d_1) + \mathcal{E}_h(d_2) \le 2\mathcal{E}_h(\frac{1}{2}|d - \tilde{d}|)$, contradicting the inequality again.) So $0 < \tilde{d} \le d$.

If $h > 1$ or $\alpha = \frac{\pi}{2}$, it is readily seen that the hypothesis of the theorem excludes all possibilities except $\tilde{d} = d$. If $h < 1$ and $\alpha \in (0, \frac{\pi}{2})$, we note that the assumption $d \in \mathbb{N} - \alpha/\pi$ implies that

$$\lim_{x_1\to\pm\infty} m_2^j(x_1) = \mp \sin\alpha$$

for every $j \in \mathbb{N}$. Due to the construction, $\hat{m}^j$ agrees with m^j in $[2j, \infty)$ for $j > 0$ and in $(-\infty, -2j]$ for $j < 0$, respectively; therefore, $\lim_{x_1\to\pm\infty} \hat{m}_2^{\pm j}(x_1) = \mp\sin\alpha$ as well, and it follows that $\frac{1}{2}(d - \tilde{d}) \in \mathbb{Z} + \{0, -\alpha/\pi\}$. Thus in this case as well, all possibilities are excluded except $\tilde{d} = d$. ☐

7 Proofs of the main results

7.1 Proof of Theorem 1.1

For the proof of Theorem 1.1, it now suffices to show that the strict inequalities required for Theorem 6.1 are satisfied in the relevant situation.

Theorem 7.1. *Suppose that $h > 1$ and $d_1, d_2 \in \mathbb{N}$ are such that E_h attains its infima in $\mathcal{A}_h(d_1)$ and in $\mathcal{A}_h(d_2)$. Then $\mathcal{E}_h(d_1 + d_2) < \mathcal{E}_h(d_1) + \mathcal{E}_h(d_2)$.*

Proof. Let $\epsilon > 0$. Suppose that $m^1 \in \mathcal{A}_h(d_1)$ and $m^2 \in \mathcal{A}_h(d_2)$ are such that

$$E_h(m^1) = \mathcal{E}_h(d_1) \quad \text{and} \quad E_h(m^2) = \mathcal{E}_h(d_2).$$

Then by Theorem 5.1 and Proposition 2.1, there exist $a, b, c > 0$ such that for any $R > a$, we can construct $\tilde{m}^1 \in \mathcal{A}_h(d_1)$ and $\tilde{m}^2 \in \mathcal{A}_h(d_2)$ with $\tilde{m}_1^1 \equiv 1$ and $\tilde{m}_1^2 \equiv 1$ outside of $(-2R, 2R)$ and with

$$E_h(\tilde{m}^1) \le \mathcal{E}_h(d_1) + ce^{-bR} \quad \text{and} \quad E_h(m^2) \le \mathcal{E}_h(d_2) + ce^{-bR}.$$

Since $d_1 \neq 0$ and $d_2 \neq 0$, by Lemma 2.4, there exists a universal constant $C_1 > 0$ such that

$$\|1 - \tilde{m}_1^1\|_{L^1(\mathbb{R})} \ge C_1 \quad \text{and} \quad \|1 - \tilde{m}_1^2\|_{L^1(\mathbb{R})} \ge C_1.$$

Suppose that $\tilde{m}^1 = (\cos\tilde{\phi}_1, \sin\tilde{\phi}_1)$ and $\tilde{m}^2 = (\cos\tilde{\phi}_2, \sin\tilde{\phi}_2)$ with $\tilde{\phi}_1(x_1) = 0$ for $x_1 \geq 2R$ and $\tilde{\phi}_2(x_1) = 0$ for $x_1 \leq -2R$. Then we define

$$\phi(x_1) = \begin{cases} \tilde{\phi}_1(x_1 + 6R) & \text{if } x_1 < 0, \\ \tilde{\phi}_2(x_1 - 6R) & \text{if } x_1 \geq 0. \end{cases}$$

Let $m = (\cos\phi, \sin\phi)$. Then $\deg(m) = d_1 + d_2$, and we have

$$\int_{-\infty}^{\infty} \left(|m'|^2 + 2W(m)\right) dx = \int_{-\infty}^{\infty} \left(|(\tilde{m}^1)'|^2 + |(\tilde{m}^2)'|^2 + 2W(\tilde{m}^1) + 2W(\tilde{m}^2)\right) dx.$$

Let $\tilde{u}_1 = U(\tilde{m}^1)$ and $\tilde{u}_2 = U(\tilde{m}^2)$ defined as in (7). Furthermore, let $u = U(m)$. Then, by the uniqueness of $U(m)$, we have $u(x) = \tilde{u}_1(x_1 + 6R, x_2) + \tilde{u}_2(x_1 - 6R, x_2)$ for $x \in \mathbb{R}^2_+$. We also have

$$\int_{\mathbb{R}^2_+} |\nabla u|^2 \, dx = \int_{\mathbb{R}^2_+} \left(|\nabla \tilde{u}_1|^2 + |\nabla \tilde{u}_2|^2\right) dx + 2\int_{\mathbb{R}^2_+} \nabla \tilde{u}_1(x_1 + 6R, x_2) \cdot \nabla \tilde{u}_2(x_1 - 6R, x_2) \, dx.$$

By Lemma 4.3, we have

$$\int_{\mathbb{R}^2_+} \nabla \tilde{u}_1(x_1 + 6R, x_2) \cdot \nabla \tilde{u}_2(x_1 - 6R, x_2) \, dx \overset{(18)}{\leq} -\frac{\|1 - \tilde{m}^1_1\|_{L^1(\mathbb{R})} \|1 - \tilde{m}^2_1\|_{L^1(\mathbb{R})}}{256\pi R^2} \leq -\frac{C_1^2}{256\pi R^2}.$$

Hence

$$E_h(m) \leq E_h(\tilde{m}^1) + E_h(\tilde{m}^2) - \frac{C_1^2}{256\pi R^2} \leq \mathcal{E}_h(d_1) + \mathcal{E}_h(d_2) + 2ce^{-bR} - \frac{C_1^2}{256\pi R^2}.$$

Therefore,

$$\mathcal{E}_h(d_1 + d_2) \leq \mathcal{E}_h(d_1) + \mathcal{E}_h(d_2) + 2ce^{-bR} - \frac{C_1^2}{16\pi R^2}.$$

For R sufficiently large, this yields the desired inequality. $\square$

Proof of Theorem 1.1. It suffices to consider $d \in \mathbb{N}$; indeed, for $d = 0$, a constant configuration will minimise E_h in $\mathcal{A}_h(0)$ and the case $d \in -\mathbb{N}$ is reduced to $d \in \mathbb{N}$ by a change of orientation.

We prove the statement by induction. For $d = 1$, it follows from the symmetrisation arguments of Melcher [15] and Chermisi-Muratov [2] that a minimiser exists. Now suppose that minimisers exist in $\mathcal{A}_h(d')$ for any $d' = 1, \ldots, d-1$. Then Theorem 7.1 implies that

$$\mathcal{E}_h(d) < \mathcal{E}_h(d') + \mathcal{E}_h(d - d')$$

for $d' = 1, \ldots, d-1$. It follows that the hypothesis of Theorem 6.1 is satisfied and that $\mathcal{E}_h$ is attained in $\mathcal{A}_h(d)$. $\square$

7.2 Proof of Theorem 1.3

Similarly to the previous section, the following strict inequality is the key here.

Theorem 7.2. *There exists a number $H \in [0, 1)$ such that whenever $h = \cos\alpha \in [H, 1)$,*

$$\mathcal{E}_h(2 - \alpha/\pi) < 2\mathcal{E}_h(1 - \alpha/\pi) + \mathcal{E}_h(\alpha/\pi).$$

Proof. Let $m^\sharp \in \mathcal{A}_h(\alpha/\pi)$ be a minimiser as in Theorem 5.2 and let $m^\flat \in \mathcal{A}_h(1 - \alpha/\pi)$ be a minimiser as in Theorem 5.3. Set $f^\sharp = m^\sharp_1 - h$ and $f^\flat = m^\flat_1 - h$. Then $f^\sharp \geq 0$ and $f^\flat \leq 0$. We have

$$\|f^\sharp\|_{L^1(\mathbb{R})} \leq C_1 \alpha$$

for a universal constant C_1 by Theorem 5.2. Furthermore, by the decay established in this theorem, we may apply Proposition 2.1 to $m^\sharp$ with three functions ω, σ, τ that satisfy

$$\omega(x_1) \leq \frac{C_2}{x_1^2}, \quad \sigma(x_1) \leq \frac{C_2}{|x_1|^3}, \quad \tau(x_1) \leq \frac{C_2}{x_1^2},$$

for $x_1 \geq c/\alpha$, where $c, C_2 > 0$ are universal constants. Hence for any $R > 2c/\alpha$ there exists a constant C_3 (possibly depending on h, but on nothing else) such that there is a map $\tilde{m}^\sharp \in \mathcal{A}_h(\alpha/\pi)$ with $\tilde{m}^\sharp_1 = \cos\alpha$ outside of $[-R, R]$ and

$$E_h(\tilde{m}^\sharp) \leq \mathcal{E}_h(\alpha/\pi) + \frac{C_3}{R^3}.$$

Furthermore, the function $\tilde{f}^\sharp = \tilde{m}_1^\sharp - h \geq 0$ still satisfies

$$\|\tilde{f}^\sharp\|_{L^1(\mathbb{R})} \leq C_1\alpha. \tag{45}$$

Similarly, there exists a map $\tilde{m}^\flat \in \mathcal{A}_h(1-\alpha/\pi)$ such that $\tilde{m}_1^\flat = h$ outside of $[-R, R]$ and

$$E_h(\tilde{m}^\flat) \leq \mathcal{E}_h(1-\alpha/\pi) + \frac{C_4}{R^3},$$

where $C_4 = C_4(h)$.

Since $m^\flat \in \mathcal{A}_h(1-\alpha/\pi)$ is symmetric, we have $m^\flat(0) = -1$ (as a complex number on $\mathbb{S}^1$). By Lemma 2.2, there exists a universal constant C_5 such that $E_h(m^\flat) \leq C_5$. Thus we obtain a universal bound for $m_1^\flat - h$ in $H^1(\mathbb{R})$ and therefore in $C^{0,1/2}([-1,1])$. The same is true for $\tilde{m}_1^\flat - h$ whenever $\alpha \leq c$ (as $R > 2$), because in this case, the two functions agree in $[-1,1]$. It follows that for $\tilde{f}^\flat = \tilde{m}_1^\flat - h \leq 0$, we have

$$\|\tilde{f}^\flat\|_{L^1(\mathbb{R})} \geq C_6 \tag{46}$$

for a universal constant $C_6 > 0$.

Define

$$m(x_1) = \begin{cases} \tilde{m}^\flat(x_1 + 4R) & \text{if } x_1 < -2R, \\ \tilde{m}^\sharp(x_1) & \text{if } |x_1| \leq 2R, \\ \tilde{m}^\flat(x_1 - 4R) & \text{if } x_1 > 2R. \end{cases}$$

Then $m \in \mathcal{A}_h(2-\alpha/\pi)$ and the arguments from the proof of Theorem 7.1 (used to compute the stray field) yield:

$$\begin{aligned} E_h(m) \leq \mathcal{E}_h(\alpha/\pi) + 2\mathcal{E}_h(1-\alpha/\pi) + \frac{C_3 + 2C_4}{R^3} \\ + \langle \tilde{f}^\flat(\,\cdot\, + 4R), \tilde{f}^\flat(\,\cdot\, - 4R)\rangle_{\dot{H}^{1/2}(\mathbb{R})} + \langle \tilde{f}^\flat(\,\cdot\, + 4R), \tilde{f}^\sharp\rangle_{\dot{H}^{1/2}(\mathbb{R})} + \langle \tilde{f}^\sharp, \tilde{f}^\flat(\,\cdot\, - 4R)\rangle_{\dot{H}^{1/2}(\mathbb{R})}. \end{aligned}$$

By Lemma 4.3 and (46), we have a universal constant C_7 such that

$$\langle \tilde{f}^\flat(\,\cdot\, + 4R), \tilde{f}^\flat(\,\cdot\, - 4R)\rangle_{\dot{H}^{1/2}(\mathbb{R})} \leq -\frac{C_7}{R^2}.$$

On the other hand, using (45), we obtain another universal constant C_8 with

$$\langle \tilde{f}^\flat(\,\cdot\, + 4R), \tilde{f}^\sharp\rangle_{\dot{H}^{1/2}(\mathbb{R})} + \langle \tilde{f}^\sharp, \tilde{f}^\flat(\,\cdot\, - 4R)\rangle_{\dot{H}^{1/2}(\mathbb{R})} \leq \frac{C_8\alpha}{R^2}.$$

Hence

$$E_h(m) \leq \mathcal{E}_h(\alpha/\pi) + 2\mathcal{E}_h(1-\alpha/\pi) + \frac{C_3 + 2C_4}{R^3} + \frac{C_8\alpha - C_7}{R^2}.$$

If we choose α small enough (i.e., H sufficiently close to 1) and R large enough, then

$$\mathcal{E}_h(2-\alpha/\pi) \leq E_h(m) < \mathcal{E}_h(\alpha/\pi) + 2\mathcal{E}_h(1-\alpha/\pi),$$

as required. □

Proof of Theorem 1.3. This is now a direct consequence of Theorem 7.2 and Theorem 6.1. □

7.3 Proof of Theorem 1.2

The statement of Theorem 1.2 is an immediate consequence of Proposition 2.3 and the following.

Lemma 7.1. *If $h < 1$, then for any $m \in \mathcal{A}_h(1)$,*

$$E_h(m) > \mathcal{E}_h(\alpha/\pi) + \mathcal{E}_h(1-\alpha/\pi).$$

Proof. Define $m_1^+ = \max\{m_1, k\}$ and $m_1^- = \min\{m_1, k\}$. Then clearly there exist $a_+, a_- \in \mathbb{R}$ such that $m_1^+(a_+) = 1$ and $m_1^-(a_-) = -1$. Thus by Lemma 1.1, there exist $m_2^\pm \colon \mathbb{R} \to [-1,1]$ such that $m^+ = (m_1^+, m_2^+) \in \mathcal{A}_h(\alpha/\pi)$ and $m^- = (m_1^-, m_2^-)$ in $\mathcal{A}_h(1-\alpha/\pi)$. Moreover,

$$\int_{-\infty}^{\infty} \left(\frac{1}{2}|m'|^2 + W(m)\right) dx_1 = \int_{-\infty}^{\infty} \left(\frac{1}{2}|(m^+)'|^2 + W(m^+) + \frac{1}{2}|(m^-)'|^2 + W(m^-)\right) dx_1.$$

Hence Lemma 4.1 implies that $E_h(m) > E_h(m^+) + E_h(m^-) \geq \mathcal{E}_h(1-\alpha/\pi) + \mathcal{E}_h(\alpha/\pi)$. □

Appendix. Nonexistence of critical points in a local model

In order to highlight the role of the nonlocal term for the existence of minimisers (or even critical points) carrying a winding number $d \geq 1$ for our variational problem, we discuss the corresponding model without the nonlocal term. For $h \geq 0$, $h \neq 1$, we consider the following Allen-Cahn type energy defined for $\phi : \mathbb{R} \to \mathbb{R}$ (representing the angle of an $\mathbb{S}^1$-valued transition layer $m = (\cos\phi, \sin\phi)$):

$$F_h(\phi) = \frac{1}{2}\int_{-\infty}^{\infty} \left((\phi')^2 + 2W(\phi)\right)\, dt.$$

Here we use the same potential W as in (1). That is, $W(\phi) = \frac{1}{2}(\cos\phi - h)^2$ if $h < 1$ and $W(\phi) = \frac{1}{2}(2h - 1 - \cos\phi)(1 - \cos\phi)$ if $h > 1$.

The Euler-Lagrange equation associated to a critical point ϕ of F_h is now given by

$$\phi'' = W'(\phi) \quad \text{in } \mathbb{R}. \tag{47}$$

Denote again $\alpha = \arccos\min\{h, 1\} \in [0, \frac{\pi}{2}]$. We impose the following boundary condition at infinity:

$$\phi(\pm\infty) := \lim_{t\to\pm\infty} \phi(t) \in 2\pi\mathbb{Z} + \{-\alpha, \alpha\}.$$

The following is well known, but we give a proof for completeness.

Theorem 7.3. *Suppose that $\phi : \mathbb{R} \to \mathbb{R}$ is a non-constant solution of equation* (47) *with boundary condition $\phi(\pm\infty) \in 2\pi\mathbb{Z} + \{-\alpha, \alpha\}$. Let $d = \frac{1}{2\pi}(\phi(+\infty) - \phi(-\infty))$ be the winding number corresponding to ϕ. If $h > 1$, then one has $d = \pm 1$. If $h < 1$, then one has $d = \pm\alpha/\pi$ or $d = \pm(1-\alpha/\pi)$.*

Proof. First, note that every solution ϕ of (47) satisfies

$$(\phi')^2 - 2W(\phi) = q \quad \text{in } \mathbb{R}$$

for some constant $q \in \mathbb{R}$. We want to prove that $q = 0$. Indeed, as ϕ has finite limits at infinity, the above equation implies that $\phi'(\pm\infty) = \ell_\pm$ for some $\ell_\pm \in \mathbb{R}$. It is enough to prove that $\ell_\pm = 0$. For this purpose, consider $X = (\phi, \phi')$ and note that X solves the following system of ODEs,

$$X' = V(X), \tag{48}$$

generated by the vector field $V(X) = (X_2, W'(X_1))$. Since $t \mapsto X(t)$ stays confined in a compact set of $\mathbb{R}^2$ and has a limit point as $t \to \pm\infty$ (by our boundary conditions for the solution ϕ), this limit point is a critical point of the vector field V, i.e., we have $X' = (0, 0)$. This implies that $\ell_\pm = 0$, and thus, that $q = 0$. In particular, the trajectory $\{X(t) = (\phi(t), \phi'(t))\}_{t\in\mathbb{R}}$ is included in the zero set of the Hamiltonian

$$H(X_1, X_2) = \frac{1}{2}X_2^2 - W(X_1), \quad X \in \mathbb{R}^2.$$

We denote $Z^\pm = \{(X_1, X_2)\colon \pm X_2 > 0, H(X_1, X_2) = 0\}$ and $Z^0 = \{(X_1, 0)\colon H(X_1, 0) = 0\} = 2\pi\mathbb{Z} + \{\pm\alpha\}$. It is readily seen that any connected component of Z^+ and Z^- ends at two consecutive points of Z^0. Obviously, any zero of Z^0 is a stationary solution of (48). Therefore, by the uniqueness of solutions to initial value problems for (48), the trajectory $\{X(t) = (\phi(t), \phi'(t))\}_t$ begins and ends at two consecutive points in Z^0. That is, in the case $h > 1$, we have winding number ± 1, and in the case $h < 1$, we have $d = \pm\alpha/\pi$ or $d = \pm(1 - \alpha/\pi)$.

□

References

[1] Antonio Capella, Christof Melcher, and Felix Otto, *Wave-type dynamics in ferromagnetic thin films and the motion of Néel walls*, Nonlinearity **20** (2007), no. 11, 2519–2537.

[2] Milena Chermisi and Cyrill B. Muratov, *One-dimensional Néel walls under applied external fields*, Nonlinearity **26** (2013), no. 11, 2935–2950.

[3] Raphaël Côte, Radu Ignat, and Evelyne Miot, *A thin-film limit in the Landau–Lifshitz–Gilbert equation relevant for the formation of Néel walls*, J. Fixed Point Theory Appl. **15** (2014), no. 1, 241–272.

[4] Antonio DeSimone, Hans Knüpfer, and Felix Otto, *2-d stability of the Néel wall*, Calc. Var. Partial Differential Equations **27** (2006), no. 2, 233–253.

[5] Antonio DeSimone, Robert V. Kohn, Stefan Müller, and Felix Otto, *A reduced theory for thin-film micromagnetics*, Comm. Pure Appl. Math. **55** (2002), no. 11, 1408–1460.

[6] ———, *Repulsive interaction of Néel walls, and the internal length scale of the cross-tie wall*, Multiscale Model. Simul. **1** (2003), no. 1, 57–104.

[7] Eleonora Di Nezza, Giampiero Palatucci, and Enrico Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136** (2012), no. 5, 521–573.

[8] Alex Hubert and Rudolf Schäfer, *Magnetic domains: The analysis of magnetic microstructures*, Springer-Verlag, Berlin, 1998.

[9] Radu Ignat, *A Γ-convergence result for Néel walls in micromagnetics*, Calc. Var. Partial Differential Equations **36** (2009), no. 2, 285–316.

[10] Radu Ignat and Hans Knüpfer, *Vortex energy and* 360° *Néel walls in thin-film micromagnetics*, Comm. Pure Appl. Math. **63** (2010), no. 12, 1677–1724.

[11] Radu Ignat and Roger Moser, *Interaction Energy of Domain Walls in a Nonlocal Ginzburg–Landau Type Model from Micromagnetics*, Arch. Ration. Mech. Anal. **221** (2016), no. 1, 419–485.

[12] Radu Ignat and Felix Otto, *A compactness result in thin-film micromagnetics and the optimality of the Néel wall*, J. Eur. Math. Soc. (JEMS) **10** (2008), no. 4, 909–956.

[13] Elliott H. Lieb and Michael Loss, *Analysis*, second ed., Graduate Studies in Mathematics, vol. 14, American Mathematical Society, Providence, RI, 2001.

[14] P.-L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case. I*, Ann. Inst. H. Poincaré Anal. Non Linéaire **1** (1984), no. 2, 109–145.

[15] Christof Melcher, *The logarithmic tail of Néel walls*, Arch. Ration. Mech. Anal. **168** (2003), no. 2, 83–113.

[16] ———, *Logarithmic lower bounds for Néel walls*, Calc. Var. Partial Differential Equations **21** (2004), no. 2, 209–219.